4.4 $\quad V(xyz) = \alpha x + \beta y^2 + \gamma z^3$

at $x=0,\ y=0,\ z=0,\ |\bar{v}| = v_0$

This is a case for conservation of energy.

$E_1 = T + V = E_0$, where

$$E_0 = \cancelto{0}{V(0,0,0)} + \frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2$$

a) When $F = (1,1,1)$

$$E_0 = \frac{1}{2}mv^2 + V(1,1,1) \quad ; \text{ solve for } v^2$$

$$v^2 = \frac{2}{m}\left(E_0 - V(1,1,1)\right)$$

$$= \frac{2}{m}\left(\frac{1}{2}mv_0^2 - \alpha - \beta - \gamma\right) \quad \text{or}$$

$$= v_0^2 - \frac{2}{m}(\alpha+\beta+\gamma)$$

b) If $v=0$ at $F = (1,1,1)$, solve for v_0

$$v_0^2 = \frac{2}{m}(\alpha+\beta+\gamma)$$

c) $\bar{F} = -\nabla V$

$$F_x = -\frac{\partial}{\partial x}V = -\alpha \Rightarrow -\alpha = ma_x = m\frac{d^2x}{dt^2}$$

$$F_y = -\frac{\partial}{\partial y}V = -2\beta y \Rightarrow -2\beta y = ma_y = m\ddot{y}$$

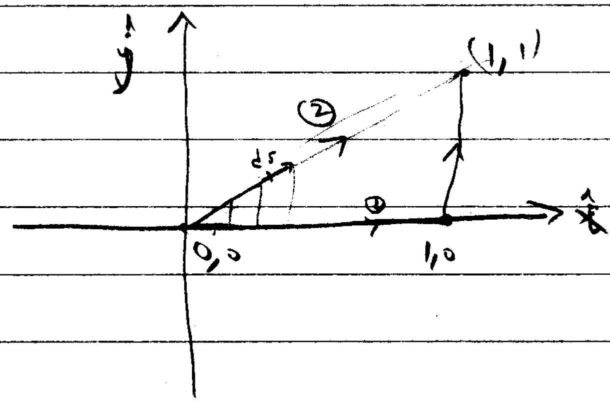
$$F_z = -\frac{\partial}{\partial z}V = -3\gamma z^2 \Rightarrow -3\gamma z^2 = ma_z = m\ddot{z}$$

4.5

We evaluate the work integral over two paths:

① $(0,0) \to (1,0) \to (1,1)$

② $(0,0) \to (1,1)$



a) $\bar{F} = \hat{\imath}\, x + \hat{\jmath}\, y$

① $$W_1 = \int_{0,0}^{1,0} \bar{F} \cdot d\bar{r} = \int_0^1 F_x\, dx + \int_0^1 F_y\, dy \qquad \frac{x^2}{2}$$

$$= \int_0^1 x\, dx + \int_0^1 y\, dy = \frac{1}{2} + \frac{1}{2} = 1$$

② Parameterize in terms of s, the path length along the line from (0,0) to (1,1), $s = \sqrt{x^2+y^2}$

$$W_2 = \int_{②} \bar{F} \cdot \frac{d\bar{r}}{ds}\, ds$$

$$= \int_{②} \left(F_x \frac{dx}{ds} + F_y \frac{dy}{ds} \right) ds$$

$$= \int_0^{\sqrt{2}} \left(s \cos^2 45 + s \sin^2 45 \right) ds$$

$$= \int_0^{\sqrt{2}} s\, ds$$

$$= 1$$

$x = s \cos 45°$

$\frac{dx}{ds} = \cos 45°$

$y = s \sin 45°$

$\frac{dx}{ds} = \sin 45°$

b) $\bar{F} = \hat{i}y - \hat{j}x$

Path 1

$$W_1 = \int_{0,0}^{1,0} \bar{F} \cdot d\bar{r}$$

$$= \int_{(0,0)}^{(1,0)} \bar{F} \cdot \frac{d\bar{r}}{dx}\, ds + \int_{1,0}^{1,1} \bar{F} \cdot \frac{d\bar{r}}{dx}\, ds$$

$$= \int_{0,0}^{1,0} y\, dx - \int_{1,0}^{1,1} x\, dy$$

From $(0,0)$ to $(1,0)$ $s = x$ so $ds = dx$;
from $(1,0)$ to $(1,1)$ $s = y$ so $ds = dy$

$$W_1 = 0 - 1 = -1$$

Path 2

$$W_2 = \int_{0,0}^{1,0} \bar{F} \cdot d\bar{r}$$

$$= \int_{0,0}^{1,0} \bar{F} \cdot \frac{d\bar{r}}{ds}\, ds =$$

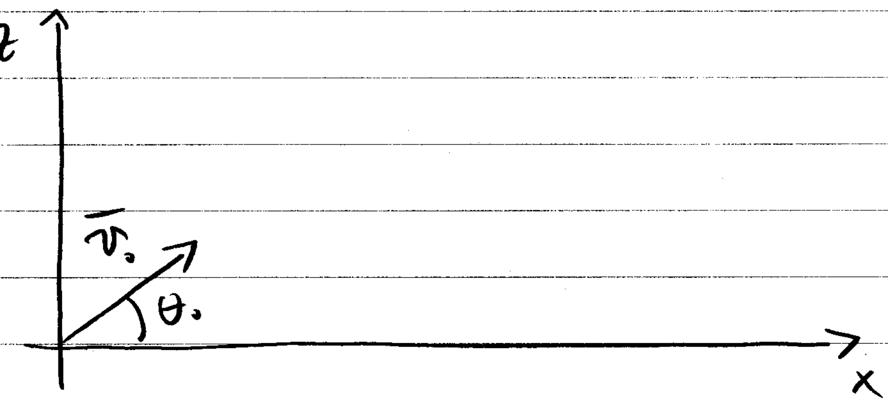
$$= \int \left(y \frac{dx}{ds} - x \frac{dy}{ds} \right) ds$$

$$= \int_0^{\sqrt{2}} \left(y \frac{1}{2} \frac{s}{x} - x \frac{1}{2} \frac{s}{y} \right) ds$$

Now $\frac{y}{x}$ and $\frac{x}{y}$ = constant = $\tan 45° = 1$. So we have $W_2 = \int_0^{\sqrt{2}} 0\, ds = 0$

4.12

~~4.9~~



With no air resistance, the max. height, h, and range, R, are

$$h = \frac{v_0^2 \sin^2\theta_0}{2g} \quad \text{and} \quad R = \frac{v_0^2 \sin 2\theta_0}{g}.$$

We have $v_0 \longrightarrow v_0 \cos\frac{\theta_0}{2}$ so our expressions are

$$h = \frac{v_0^2}{2g} \cos^2\frac{\theta_0}{2} \sin^2\theta_0 \quad \text{and} \quad R = \frac{v_0^2}{g} \cos^2\frac{\theta_0}{2} \sin 2\theta_0.$$

Now, we could take $\frac{d}{d\theta} h = 0$ and $\frac{d}{d\theta} R = 0$.

That's too messy and boring. Let's do it numerically:

θ_0 (°)	$\cos^2\frac{\theta_0}{2} \sin^2\theta_0$	$\cos^2\frac{\theta_0}{2} \sin 2\theta_0$
20	.113	.623
30	.233	.808
40	.365	.870
50	.482	.810
60	.562	.650
70	.592	.431
80	.569	.201

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So roughly speaking, we have max. h at $\theta_0 \approx 70°$ and max. R at $\theta_0 \approx 40°$.

At $\theta_0 = 70°$, with $v_0 = 25\,m/sec$,

$$h_{max} = \frac{(25\,m/sec)^2}{2(9.8\,m/sec^2)}(.592) = 18.9\,m$$

and
$$R_{max} = \frac{(25\,m/sec)^2}{9.8\,m/sec^2} \cdot (.870) = 55.5\,m.$$

If we try θ_0 in finer increments, we'd obtain max h at 71° and max R at 40°.

While we like to have analytical expressions, a numerical solution is also respectable when the equations are complicated.

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4.11 Air resistance proportional to the square of the speed.

2^{nd} "Law"

$$\bar{F} = m\bar{a}$$

In this case $\bar{F} = -mg\hat{k} - C_2 v^2 \hat{v}$

Note that whatever is the direction of $\bar{v}$, the drag is <u>opposite</u>. The problem is to express the $\hat{v}$ in terms of the fixed unit vectors $\hat{i}$ + $\hat{k}$. It turns out not to be difficult:

$$\hat{v} = \frac{\bar{v}}{v}$$
$$v^2\hat{v} = v\,\bar{v}$$
$$v^2\hat{v} = v\,(v_x\hat{i} + v_z\hat{k})$$

So $\bar{F} = -mg\hat{k} - C_2 v\,(v_x\hat{i} + v_z\hat{k})$,
where $v = |\bar{v}|$.

Components:

x: $\ddot{x} = -\frac{C_2 v}{m} v_x = -\frac{C_2 v}{m}\dot{x} = -\frac{C_2 \dot{s}\dot{x}}{m}$

z: $\ddot{z} = -g - \frac{C_2 v}{m} v_z = -g - \frac{C_2 v}{m}\dot{z}$
$= -g - \frac{C_2}{m}\dot{s}\dot{z}$

These are <u>not</u> separable.

To obtain $\dot{x}$ and $\dot{z}$ we integrate...

$$\ddot{x} = -\frac{c_2}{m} \dot{s} \dot{x}$$

$$\frac{d\dot{x}}{dt} = -\frac{c_2}{m} \dot{s} \dot{x}$$

Treat s + x as separate variables,

$$\frac{d\dot{x}}{ds} \frac{ds}{dt} = -\frac{c_2}{m} \dot{s} \dot{x}$$

$$\frac{d\dot{x}}{ds} \dot{s} = -\frac{c_2}{m} \dot{s} \dot{x}$$, the $\dot{s}$ cancels!

$$\int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\frac{c_2}{m} \int_0^s ds$$

$$\ln \dot{x} = \ln \dot{x}_0 - \frac{c_2}{m} s$$

$$\dot{x} = \dot{x}_0 e^{-\frac{c_2}{m} s}$$

Notice the author didn't ask us to get $\dot{z}$ as a function of s. That's harder because of the constant g.

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~~4,8~~

Isotropic harmonic oscillator

Separated eqns of motion. (p154-155)

$$m\ddot{x} = -kx$$
$$m\ddot{y} = -ky$$

Initial condition

at $t=0$; $x = A$, $y = 4A$, $\dot{x} = 0$, $\dot{y} = 3\omega A$

Proposed solution in x

$$x = a\cos(\omega t + \alpha)$$

apply initial conditions

$$A = a\cos\alpha \; ; \; a = A$$
$$0 = -a\omega\sin\alpha \rightarrow \alpha = 0$$

Proposed solution in y

$$y = b\cos(\omega t + \beta)$$

apply initial conditions

$$4A = b\cos\beta$$
$$3\omega A = -b\omega\sin\beta$$

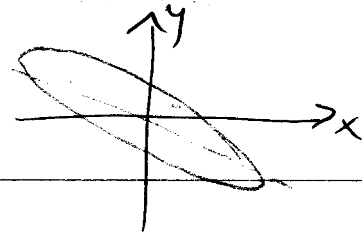
!!

divide 2nd eqn by first

$$\frac{3\omega}{4} = -\omega\tan\beta \rightarrow \tan\beta = -\frac{3}{4} \rightarrow \beta = -.64 \text{ radian}$$

Substitute back into 1st eqn

$$b = \frac{4A}{\cos\beta} = \frac{4}{.8}A = 5A$$



4.13 continued

The trajectory is given by eqn 4.4.10

$$\frac{x^2}{a^2} - xy\,\frac{2\cos(\beta-\alpha)}{ab} + \frac{y^2}{b^2} = \sin^2(\beta-\alpha)$$

We have $a = A$ and $b=5A$ and $\beta-\alpha = -.644$ radian

$$\frac{x^2}{A^2} - xy\,\frac{2(.8)}{5A^2} + \frac{y^2}{25A^2} = (-0.6)^2$$

This is the equation for an ellipse.
The angle the major axis makes with the $\hat{x}$ axis is eqn 4.4.15

$$\tan 2\psi = \frac{2ab\cos(\beta-\alpha)}{a^2-b^2}$$

$$\tan 2\psi = \frac{10A^2\cos(-.644)}{-24A^2} = -\frac{10}{24}\cos(-.644)$$

$$\tan 2\psi = -\frac{10}{24}(.8) = -0.333$$

$$\psi = \tfrac{1}{2}\tan^{-1}(-0.333) = -.16153 \text{ rad.} \approx -9.2^\circ$$

The amplitudes of $A + 5A$ will define a rectangle of sides $2A \times 10A$.

4.16 alternative

$$x = A_1 \sin\omega t + A_2 \cos\omega t$$
$$\dot{x} = \omega A_1 \cos\omega t - \omega A_2 \sin\omega t$$

$t=0 \quad x_0 = A \quad \dot{x}_0 = 0$

$$A = A_1 \cdot 0 + A_2 \cdot 1 \Rightarrow A_2 = A$$
$$0 = \omega A_1 \cdot 1 - \omega A_2 \cdot 0 \Rightarrow A_1 = 0$$

$$x = A \cos\omega t$$

$$y = B_1 \sin\omega t + B_2 \cos\omega t$$
$$\dot{y} = \omega B_1 \cos\omega t - \omega B_2 \sin\omega t$$

$t=0 \quad y_0 = 4A \quad \dot{y}_0 = 3\omega A$

$$4A = B_1 \cdot 0 + B_2 \cdot 1 \Rightarrow B_2 = 4A$$
$$3\omega A = \omega B_1 \cdot 1 - \omega B_2 \cdot 0 \Rightarrow B_1 = 3A$$

$$y = 3A \sin\omega t + 4A \cos\omega t$$

The limits along the x-axis are $-A$ to $+A$.
The limits along the y-axis are $-5A$ to $+5A$.
$(3^2 + 4^2 = 25 = 5^2)$
So the oscillator is confined to a rectangle of dimension $2A \times 10A$.

See equation 4.4.15 for the angle of the ellipse to the x-axis.