CHAPTER 6
GRAVITATIONAL AND CENTRAL FORCES

6.1 $m = \rho V = \rho \frac{4}{3}\pi r_s^3$

$$r_s = \left(\frac{3m}{4\pi\rho}\right)^{\frac{1}{3}}$$

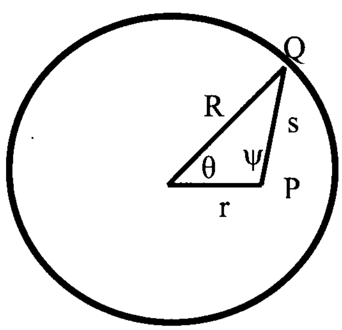
$$F = \frac{Gmm}{(2r_s)^2} = \frac{Gm^2}{4}\left(\frac{4\pi\rho}{3m}\right)^{\frac{2}{3}} = \frac{G}{4}\left(\frac{4\pi\rho}{3}\right)^{\frac{2}{3}} m^{\frac{4}{3}}$$

$$\frac{F}{W} = \frac{F}{mg} = \frac{Gm^2}{4g}\left(\frac{4\pi\rho}{3}\right)^{\frac{2}{3}} m^{\frac{1}{3}}$$

$$\frac{F}{W} = \frac{6.672\times10^{-11} N\cdot m^2\cdot kg^{-2}}{4\times9.8\,m\cdot s^{-2}}\left(\frac{4\pi\times11.35\,g\cdot cm^{-3}}{3}\times\frac{1kg}{10^3 g}\times\frac{10^6 cm^3}{1m^3}\right)^{\frac{2}{3}}\times(1kg)^{\frac{1}{3}}$$

$$\frac{F}{W} = 2.23\times10^{-9}$$

6.2 (a) The derivation of the force is identical to that in Section 6.2 except here r < R. This means that in the last integral equation, (6.2.7), the limits on u are R – r to R + r.



$$F = \frac{GmM}{4Rr^2}\int_{R-r}^{R+r}\left(1+\frac{r^2-R^2}{s^2}\right)ds$$

$$= \frac{GmM}{4Rr^2}\left[R+r-(R-r)+\frac{R^2-r^2}{R+r}-\frac{R^2-r^2}{R-r}\right]$$

$$F = \frac{GmM}{4Rr^2}\left[2r+R-r-(R+r)\right]=0$$

(b) Again the derivation of the gravitational potential energy is identical to that in Example 6.7.1, except that the limits of integration on s are $(R-r)\to(R+r)$.

$$\phi = -G\frac{2\pi\rho R^2}{rR}\int_{R-r}^{R+r} ds$$

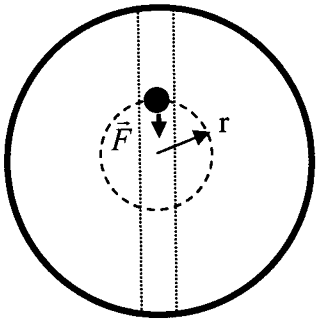
$$= -G\frac{2\pi\rho R^2}{rR}\left[R+r-(R-r)\right]$$

$$\phi = -G\frac{4\pi R^2\rho}{R} = -G\frac{M}{R}$$

For $r < R$, ϕ is independent of r. It is constant inside the spherical shell.

6.3 $\vec{F} = -\frac{GMm}{r^2}\hat{e}_r$

The gravitational force on the particle is due only to the mass of the earth that is inside the particle's instantaneous displacement from the center of the earth, r. The net effect of the mass of the earth outside r is zero (See Problem 6.2).



$$M = \frac{4}{3}\pi r^3 \rho$$

$$\vec{F} = -\frac{4}{3}G\pi\rho m r\hat{e}_r = -kr\hat{e}_r$$

The force is a linear restoring force and induces simple harmonic motion.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3}{4G\pi\rho}}$$

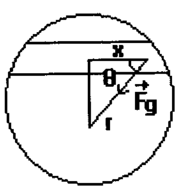
The period depends on the earth's density but is independent of its size.
At the surface of the earth,

$$mg = \frac{GMm}{R_e^2} = \frac{Gm}{R_e^2}\cdot\frac{4}{3}\pi R_e^3\rho$$

$$\frac{4G\pi\rho}{3} = \frac{g}{R_e}$$

$$T = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6.38\times10^6 m}{9.8 m\cdot s^{-2}}}\times\frac{1 hr}{3600 s} \approx 1.4 hr$$

6.4



$$\vec{F}_g = -\frac{GMm}{r^2}\hat{e}_r \text{, where } M = \frac{4}{3}\pi r^3 \rho$$

The component of the gravitational force perpendicular to the tube is balanced by the normal force arising from the side of the tube. The component of force along the tube is

$$F_x = F_g \cos\theta$$

The net force on the particle is …

$$\vec{F} = -\hat{i}\frac{4}{3}G\pi\rho m r\cos\theta$$

$$r\cos\theta = x$$

$$\vec{F} = -\hat{i}\frac{4}{3}G\pi\rho m x = -\hat{i}kx$$

As in problem 6.3, the motion is simple harmonic with a period of 1.4 hours.

6.5 $$\frac{GMm}{r^2}=\frac{mv^2}{r} \quad \text{so} \quad v^2=\frac{GM}{r}$$

for a circular orbit r, v is constant.

$$T=\frac{2\pi r}{v}$$

$$T^2=\frac{4\pi^2 r^2}{v^2}=\frac{4\pi^2}{GM}r^3 \propto r^3$$

6.6 (a) $T=\dfrac{2\pi r}{v}$

From Example 6.5.3, the speed of a satellite in circular orbit is …

$$v=\left(\frac{gR_e^2}{r}\right)^{\frac{1}{2}}$$

$$T=\frac{2\pi r^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}$$

$$r=\left(\frac{T^2 gR_e^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\frac{r}{R_e}=\left(\frac{T^2 g}{4\pi^2 R_e}\right)^{\frac{1}{3}}=\left(\frac{24^2\,hr^2\times 3600^2\,s^2\cdot hr^{-2}\times 9.8\,m\cdot s^{-2}}{4\pi^2 6.38\times 10^6\,m}\right)^{\frac{1}{3}}$$

$$\frac{r}{R_e}=6.62\approx 7$$

(b) $$T=\frac{2\pi r^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}=\frac{2\pi(60R_e)^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}=2\pi\sqrt{\frac{60^3 R_e}{g}}$$

$$=2\pi\left(\frac{60^3\times 6.38\times 10^6\,m}{9.8\,m\cdot s^{-2}\times 3600^2\,s^2\cdot hr^{-2}\times 24^2\,hr^2\cdot day^{-2}}\right)^{\frac{1}{2}}$$

$$T=27.27\,day\approx 27\,day$$

6.7 From Example 6.5.3, the speed of a satellite in a circular orbit just above the earth's surface is …

$$v=\sqrt{gR_e}$$

$$T=\frac{2\pi R_e}{v}=2\pi\sqrt{\frac{R_e}{g}}$$

$$M_d = \frac{4}{3}\pi r^3 \rho$$

$$F(r) = -\frac{GMm}{r^2} - \frac{4}{3}\pi\rho m G r$$

6.10 $u = \frac{1}{r} = \frac{1}{r_\circ} e^{-k\theta}$

$$\frac{du}{d\theta} = -\frac{k}{r_\circ} e^{-k\theta}$$

$$\frac{d^2u}{d\theta^2} = \frac{k^2}{r_\circ} e^{-k\theta} = k^2 u$$

From equation 6.5.10 …

$$\frac{d^2u}{du^2} + u = k^2 u + u = -\frac{1}{ml^2u^2} f\left(u^{-1}\right)$$

$$f\left(u^{-1}\right) = -ml^2\left(k^2+1\right)u^3$$

$$f(r) = -\frac{ml^2\left(k^2+1\right)}{r^3}$$

The force varies as the inverse cube of r.

From equation 6.5.4, $r^2\dot{\theta} = l$

$$\frac{d\theta}{dt} = \frac{l}{r_\circ^2} e^{-2k\theta}$$

$$e^{2k\theta} d\theta = \frac{l}{r_\circ^2} dt$$

$$\frac{1}{2k} e^{2k\theta} = \frac{lt}{r_\circ^2} + C$$

$$\theta = \frac{1}{2k} \ln\left(\frac{2klt}{r_\circ^2} + C'\right)$$

θ varies logarithmically with t.

6.11 $f(r) = \frac{k}{r^3} = ku^3$

From equation 6.5.10 …

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} \cdot ku^3 = -\frac{ku}{ml^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{k}{ml^2}\right)u = 0$$

If $\left(1+\frac{k}{ml^2}\right)<0$, $\quad \frac{d^2u}{d\theta^2}-cu=0$, $\quad c>0$, for which $u=ae^{b\theta}$ is a solution.

If $\left(1+\frac{k}{ml^2}\right)=0$, $\quad \frac{d^2u}{d\theta^2}=0$

$$\frac{du}{d\theta}=C_1$$

$$u=c_1\theta+c_2$$

$$r=\frac{1}{c_1\theta+c_2}$$

If $\left(1+\frac{k}{ml^2}\right)>0$, $\quad \frac{d^2u}{d\theta^2}+cu=0$, $\quad c>0$

$$u=A\cos\left(\sqrt{c}\theta+\delta\right)$$

$$r=\left[A\cos\left(\sqrt{1+\frac{k}{ml^2}}\theta+\delta\right)\right]^{-1}$$

6.12 $u=\frac{1}{r}=\frac{1}{r_\circ\cos\theta}$

$$\frac{du}{d\theta}=\frac{\sin\theta}{r_\circ\cos^2\theta}$$

$$\frac{d^2u}{d\theta^2}=\frac{1}{r_\circ}\left(\frac{1}{\cos\theta}+\frac{2\sin^2\theta}{\cos^3\theta}\right)=\frac{1}{r_\circ\cos\theta}\left(1+\frac{2-2\cos^2\theta}{\cos^2\theta}\right)=\frac{1}{r_\circ\cos\theta}\left(\frac{2}{\cos^2\theta}-1\right)$$

$$\frac{d^2u}{d\theta^2}=u\left(2r_\circ^2u^2-1\right)=2r_\circ^2u^3-u$$

Substituting into equation 6.5.10 …

$$2r_\circ^2u^3-u+u=-\frac{1}{ml^2u^2}f\left(u^{-1}\right)$$

$$f\left(u^{-1}\right)=-2r_\circ^2ml^2u^5$$

$$f\left(r\right)=-\frac{2r_\circ^2ml^2}{r^5}$$

6.13 From Chapter 1, the transverse component of the acceleration is … $a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}$

If this term is nonzero, then there must be a transverse force given by …

$$f(\theta)=m(r\ddot{\theta}+2\dot{r}\dot{\theta})$$

For $\quad r=a\theta$, and $\quad \theta=bt$

$$f\left(\theta\right)=2mab^2\neq0$$

Since $f\left(\theta\right)\neq0$, the force is not a central field.

$$E = -\frac{k}{2a} \qquad \text{So} \qquad \delta E = \frac{k}{2a^2}\delta a$$

Since planetary orbits are nearly circular

$$V \sim -\frac{k}{a} \qquad \text{and} \qquad T \sim \frac{k}{2a}$$

Thus, $\delta E \cong T\frac{\delta a}{a}$ and $\frac{\delta E}{T} = \frac{\delta a}{a}$

We obtain $\frac{\delta a}{a} = 2\frac{\delta v}{v}$

6.20 (a) $\bar{V} = \frac{1}{\tau}\int_0^\tau V dt$

$$V(r) = -\frac{k}{r}$$

From equation 6.5.4, $l = r^2\dot{\theta}$

$$\frac{d\theta}{dt} = \frac{l}{r^2} \text{ or } dt = \frac{r^2 d\theta}{l}$$

$$\int_0^\tau V dt = -\int_0^{2\pi}\frac{kr}{l}d\theta$$

From equation 6.5.18a ...

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$$

$$\int_0^\tau V dt = -\frac{ka(1-\varepsilon^2)}{l}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

From equation 6.6.4 ... $\tau = \frac{2\pi a^2}{l}\sqrt{1-\varepsilon^2}$

$$\bar{V} = -\frac{k\sqrt{1-\varepsilon^2}}{2\pi a}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

$$\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}, \ \varepsilon^2 < 1 \ \therefore \bar{V} = -\frac{k}{a}$$

(b) This problem is an example of the *virial theorem* which, for a bounded, periodic system, relates the time average of the quantity $\int_0^\tau \sum_i \bar{p}\cdot\bar{r}_i$ to its kinetic energy T. We will derive it for planetary motion as follows:

$$\frac{1}{\tau}\int_0^\tau \bar{\dot{p}}\cdot\bar{r}dt = \frac{1}{\tau}\int_0^\tau m\bar{\ddot{r}}\cdot\bar{r}dt = \frac{1}{\tau}\int_0^\tau \bar{F}\cdot\bar{r}dt$$

Integrate LHS by parts

$$\frac{1}{\tau}\left[m\dot{\bar{r}}\cdot\bar{r}\right]\Big|_0^\tau-\frac{1}{\tau}\int_0^\tau m\dot{\bar{r}}^2dt=\frac{1}{\tau}\int_0^\tau \bar{F}\cdot\bar{r}dt$$

The first term is zero – since the quantity has the same value at 0 and τ.

Thus $2\langle T\rangle=-\langle\bar{F}\cdot\bar{r}\rangle$ where $\langle\ \rangle$ denote time average of the quantity within brackets.

but $-\langle\bar{r}\cdot\bar{F}\rangle=\langle r\cdot\bar{\nabla}V\rangle=\left\langle r\frac{dV}{dr}\right\rangle=\left\langle\frac{k}{r}\right\rangle=-\langle V\rangle$

hence $2\langle T\rangle=-\langle V\rangle$

but $\langle E\rangle=\langle T\rangle+\langle V\rangle=-\frac{\langle V\rangle}{2}+\langle V\rangle=\frac{\langle V\rangle}{2}$

hence $\langle V\rangle=2\langle E\rangle$ but $E=-\frac{k}{2a}=constant$

and $\langle E\rangle=\frac{1}{\tau}\int_0^\tau Edt=E=-\frac{k}{2a}$ so $2E=-\frac{k}{a}$

Thus: $\langle V\rangle=-\frac{k}{a}$ as before and therefore $\langle T\rangle=-\frac{1}{2}\langle V\rangle=\frac{k}{2a}$

6.21 The energy of the initial orbit is

$$\frac{1}{2}mv^2-\frac{k}{r}=E=-\frac{k}{2a}$$

(1) $$v^2=\frac{k}{m}\left(\frac{2}{r}-\frac{1}{a}\right)$$

Since $r_a=a(1+\varepsilon)$ at apogee, the speed v_1, at apogee is

$$v_1^2=\frac{k}{m}\left(\frac{2}{a(1+\varepsilon)}-\frac{1}{a}\right)=\frac{k}{ma}\frac{(1-\varepsilon)}{(1+\varepsilon)}$$

To place satellite in circular orbit, we need to boost its speed to v_c such that

$$\frac{1}{2}mv_c^2-\frac{k}{r_a}=-\frac{k}{2r_a}$$ since the radius of the orbit is r_a

$$v_c^2=\frac{k}{mr_a}=\frac{k}{ma(1+\varepsilon)}$$

Thus, the boost in speed $\Delta v_1=v_c-v_1$

(2) $$\Delta v_1=\left[\frac{k}{ma(1+\varepsilon)}\right]^{\frac{1}{2}}\left[1-(1-\varepsilon)^{\frac{1}{2}}\right]$$

Now we solve for the semi-major axis a and the eccentricity ε of the first orbit. From (1) above, at launch $v=v_\circ$ at $r=R_E$, so

$$v_\circ^2=\frac{k}{m}\left(\frac{2}{R_E}-\frac{1}{a}\right)$$