I. Equations of Motion

A. Linear First Order Ordinary Differential Equation

1. Newton's 2nd law

Applied to a particle in one dimension,

\[ m \frac{d^2x}{dt^2} = F(t) \]

We get an initial value problem:

\[ \frac{d^2x}{dt^2} = f(x, t) \]

with \((x_0, t_0)\) given. Find \(x(t)\) that passes through \((x_0, t_0)\) and has \(x'(t) = f(x, t)\).

Such a problem may be solved analytically by integration or separation of variables or etc., sometimes that's too hard or otherwise inconvenient.

2. Numerical Solution

A numerical solution consists of a set of points \((x_i, t_i), (x_i, t_{i+1}), (x_i, t_{i+2}), \ldots, (x_i, t_n)\) which approximate \(x(t_i), x(t_{i+1}), x(t_{i+2}), \ldots, x(t_n)\).
1. The trapezoidal rule

The trapezoidal rule is approximated by integrating the function to the function.

We have:

\[ x_i = \frac{1}{h} \left( x_i + x_i^{(i-1)} \right) \]

This method is the Taylor series.

The result is a more accurate polynomial approximation.
Thus

\[ x' = f \]
\[ x'' = f' = f_t + f_x f \]
\[ x''' = f'' = f_{tt} + 2f_{tx} + f^2 f_x + f_x [f_t + f_x f] \]

As you may imagine, we normally take \( p \) to be small.

c. Euler's Method, \( p = 1 \)

\[ x(t_{i+1}) = x(t_i) + h f(x(t_i), t_i) + T_i \]

The numerical algorithm is

\[ x_{i+1} = x_i + h f(x_i, t_i) + T_i. \]

There is truncation error (T) and round-off error (R) at each step. These accumulate from step to step.

![Graph of x(t) with points (x_i, t_i) and (x_i+1, t_i+1)]
What to do to reduce $T$? Use larger $p \rightarrow$ Runge-Kutta methods. Use smaller $h$, as illustrated in chap 1.

d. At each step, the quantity $f(x, t)$ has to be evaluated. One would like to evaluate that as few times as possible, meaning large $h$. But, the larger $h$ is, the greater $T$ will be. So to minimize $T$ one would choose a small $h$.

Obviously, a compromise must be made. That compromise is determined by trial and error — repeating a calculation with varied $h$ to find the largest one that gives acceptable $T$. 