I. INTRODUCTION ................................................................................................................................. 3

A. Fundamental Concepts and Assumptions ......................................................................................... 3
   1. Concepts ........................................................................................................................................... 3
   2. Assumptions ..................................................................................................................................... 3

B. Kinematics – Describing Motion ...................................................................................................... 3
   1. Coordinate Systems ......................................................................................................................... 3
   2. Variables of Motion ......................................................................................................................... 4
   3. Galilean Relativity ......................................................................................................................... 5

C. Newton’s “Laws” of Motion ............................................................................................................. 6
   1. First “Law” or “Law” of Inertia ....................................................................................................... 6
   2. Second “Law” ................................................................................................................................. 7
   3. Third “Law” .................................................................................................................................... 7

II. DYNAMICS IN ONE DIMENSION ...................................................................................................... 8

A. Constant Force ................................................................................................................................. 8
   1. Equation of Motion ......................................................................................................................... 8
   2. Examples ......................................................................................................................................... 9

B. Force as an Explicit Function of Time ............................................................................................ 9
   1. Equation of Motion ......................................................................................................................... 9
   2. Examples ....................................................................................................................................... 10
   3. Impulse, a Vector .......................................................................................................................... 10

C. Force as a Function of Position ....................................................................................................... 11
   1. Equation of Motion in One Dimension .......................................................................................... 11
   2. Potential Energy Function ............................................................................................................ 12

D. Force as a Function of Velocity ...................................................................................................... 14
   1. Equations of Motion—Two Ways to Go ....................................................................................... 14
   2. Examples ....................................................................................................................................... 15

E. Harmonic Oscillator ....................................................................................................................... 17
   1. Simple Harmonic Oscillator in One Dimension ......................................................................... 17
   2. Damped Harmonic Oscillator ....................................................................................................... 18
   3. Driven Harmonic Oscillator ......................................................................................................... 20

III. DYNAMICS OF A POINT IN THREE DIMENSIONS ....................................................................... 24

A. Extension of the Concepts to Three Dimensions ........................................................................... 24
   1. Impulse .......................................................................................................................................... 24
   2. Work-Energy Theorem ............................................................................................................... 24
   3. Work Integrals .............................................................................................................................. 25
   4. Potential Energy Functions .......................................................................................................... 26
   5. Angular Momentum ..................................................................................................................... 27
   6. Examples ....................................................................................................................................... 28

B. Separable Forces ............................................................................................................................ 29
   1. Projectile Motion in a Uniform Gravitational Field .................................................................... 29
2. Harmonic Oscillator........................................................................................................ 34

C. Constrained Motion of a Particle.................................................................................. 34
   1. Smooth Constraints .................................................................................................... 34
   2. Motion on a Curve ................................................................................................... 35

IV. ACCELERATED REFERENCE FRAMES .................................................................. 37
   A. Galilean Transformation .......................................................................................... 37
      1. Transformation Equations ..................................................................................... 37
      2. Translating Reference Frames ............................................................................... 37
   B. Rotating Reference Frames ..................................................................................... 38
      1. Equations of Motion ............................................................................................... 38
      2. Rotating Earth ......................................................................................................... 41

V. POTPOURRI ............................................................................................................. 45
   A. Systems of Particles ............................................................................................... 45
      1. *N*-particles ........................................................................................................... 45
      2. Rocket .................................................................................................................... 46
      3. Collisions ................................................................................................................ 48
   B. Rigid Body ............................................................................................................... 49
      1. Equations of motion ............................................................................................... 49
      2. Computing moments of inertia ............................................................................. 51
      3. Laminar Motion of a Rigid Body .......................................................................... 53
   C. Central Forces ........................................................................................................ 56
      1. General Properties ................................................................................................. 56
      2. Orbits ..................................................................................................................... 57
I. Introduction

A. Fundamental Concepts and Assumptions

1. Concepts

   a. Space and time, defined operationally
      That is, space and time are defined by specifying how they are to be measured.

   b. Particle
      A particle is an object with mass but no extent, or volume, nor internal structure.

2. Assumptions

   a. Physical space can be described by 3-dimensional Euclidean geometry.

   b. An ordered sequence of events can be measured on a uniform and absolute time scale. I.e.,
      time intervals are measured the same by all observers.

   c. Time and space are distinct and independent quantities.

B. Kinematics – Describing Motion

1. Coordinate Systems

   a. Cartesian

      \( \hat{i}, \hat{j}, \text{ and } \hat{k} \) are unit vectors in a right-handed set of
      coordinate axes. That is, \( \hat{k} = \hat{i} \times \hat{j} \), etc.
b. Spherical polar

The unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are not constant.

\[\begin{align*}
\end{align*}\]

\[\begin{align*}
\end{align*}\]

\[\begin{align*}
\end{align*}\]

c. Cylindrical

The unit vectors $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ are not constant.

\[\begin{align*}
\end{align*}\]

\[\begin{align*}
\end{align*}\]

\[\begin{align*}
\end{align*}\]

2. Variables of Motion

To describe the motion of a particle, we use 4 variables: $\vec{r}$, $\vec{v}$, $\vec{a}$, and $t$; all measured with respect to some selected origin of coordinates.

a. Position, $\vec{r}$

Selection of coordinate system is arbitrary, but once chosen must be adhered to.

b. Velocity, $\vec{v}$

\[\frac{d\vec{r}}{dt}\]

c. Acceleration, $\vec{a}$

\[\frac{d^2\vec{r}}{dt^2}\]

d. Equation of motion

An equation of motion is a differential equation relating the variables of motion to one another, esp. to $t$. We have solved the equation of motion when we obtain $\vec{r}(t)$ and $\vec{v}(t)$. 

\[\begin{align*}
\end{align*}\]
e. Example – constant acceleration
\[
\ddot{r}(t) = \ddot{r} = \ddot{r}_o + \dddot{v}_o + \frac{1}{2} \dddot{a} t^2 \quad \text{and} \quad \ddot{v}(t) = \ddot{v} = \ddot{v}_o + \dddot{a}
\]

3. Galilean Relativity
The motion of a particle may be described in terms of different reference frames. How do we relate the motion variables measured in one frame to those measured in another frame?

a. Two reference frames

Viewed from O, the point P has velocity \( \vec{v} \) and acceleration \( \vec{a} \).
Relative to O’, the point P is observed to have velocity \( \vec{v}' \) and acceleration \( \vec{a}' \). In general \( \vec{v} \neq \vec{v}' \) and \( \vec{a} \neq \vec{a}' \).

b. Galilean transformation, without rotation
To see clearly how the transformation goes, let’s take a simpler case:

Note that \( \vec{r} = \vec{R} + \vec{r}' \). Then just take the time derivative: \( \ddot{v} = \frac{d\ddot{r}}{dt} = \frac{d\ddot{R}}{dt} + \dddot{r}' = \frac{d\ddot{R}}{dt} + \dddot{v}' \).

The quantity \( \frac{d\ddot{R}}{dt} \) is the \textit{relative velocity} of the two reference frames. Let’s say that \( \ddot{u} = \frac{d\ddot{R}}{dt} \).

Finally, \( \dddot{a} = \frac{d\dddot{v}}{dt} = \frac{d\dddot{u}}{dt} + \dddot{v}' = \dddot{A} + \dddot{a}' \). We have \textit{assumed} that \( dt \) is the same in both reference frames.
c. Examples
   i) boat, drifting with the current
   \[ \vec{v} = \vec{u} + \vec{v}' \]

   ii) elevator, with a person standing on a scale
   We have \( a' = 0 \) and \( a = A \). Newton’s Second “Law” says:
   \[ \sum \vec{F}_i = m \vec{a} \]. Write the vertical components out separately in the two reference frames.

   in the elevator: \( F'_c - mg' = 0 \)
   
   as seen by an observer outside the elevator: \( F'_c - mg = mA \)

   The “true” weight, or gravitational force exerted by the Earth on the person is \( mg \). In the elevator, the scale reading is interpreted as the apparent weight, \( mg' = F'_c \). Only when we compare the two viewpoints do we see that \( mg' = mg + mA \). Note, also, that \( A \) may be plus (up) or minus (down), or zero, or -g.

C. Newton’s “Laws” of Motion
To describe the relation among motion, changes in motion, and forces, Newton proposed three “Laws.”

1. First “Law” or “Law” of Inertia
A body in uniform motion remains in uniform motion unless acted upon by an external net force.

   a. Inertia
   This property of matter expressed by the First “law” is called inertia. The quantitative measure of inertia a body has is its inertial mass.

   b. Inertial reference frames
   A reference frame in which Newton’s three “Laws” are valid is called an inertial reference frame. One in which they do not hold is called noninertial, for instance, the rotating Earth.
2. Second “Law”

\[ \ddot{F} = \sum_i F_i = m\ddot{a} \]

The acceleration of a body is proportional to the net force, and in the direction of the net force.

a. Process
Identify all applied forces, \( F_i \). Add them vectorially to obtain the net force, \( \ddot{F} \). Set \( \ddot{F} = m\ddot{a} \) and solve for \( \dddot{v} \) and \( \dddot{r} \).

b. Quantitative definition of inertial mass
Let’s consider two bodies, 1 and 2, connected by a spring. Observe the motion:

\[ \dddot{a}_1 = \mu \cdot \dddot{a}_2 \]

We wish to obtain a parameter characteristic of each body, so rewrite this observation as

\[ m_1\dddot{a}_1 = -m_2\dddot{a}_2 \]

Ultimately, all masses are measured relative to some standard kilogram.

c. Translational momentum
We identify the “change in motion” caused by a force as \( m\dddot{a} = m\frac{d\dddot{v}}{dt} \).

Definition: translational momentum, \( \dddot{p} = m\dddot{v} \).

3. Third “Law”

We observed the two masses connected by a spring: \( m_1\dddot{a}_1 = -m_2\dddot{a} \).

a. Action-reaction
The force exerted by body 1 on body 2 (\( F_{12} \)) is equal and opposite to that exerted by body 2 on body 1 (\( F_{21} \)). I.e., \( F_{12} = -F_{21} \).

b. Interaction
In effect, the Third “Law” is describing a property of force: that it is an interaction between two masses. Put another way, any two bits of matter exert forces on each other.

c. Conservation of translational momentum
If two bodies are isolated, then

\[ m_1\dddot{a}_1 + m_2\dddot{a}_2 = 0 \]

\[ m_1 \frac{d\dddot{v}_1}{dt} + m_2 \frac{d\dddot{v}_2}{dt} = 0 \]

\[ m_1\dddot{v}_1 + m_2\dddot{v}_2 = a \text{ constant} \]
II. Dynamics in One Dimension

Dynamics is the application of calculus to solve equations of motion. We often begin the discussion of dynamics in the context of simplified one-dimensional situations.

A. Constant Force

1. Equation of Motion

a. Coordinate frame

\[ \vec{F} = m\ddot{a} \]

b. X-component equation of motion

\[ F_x = ma_x. \]

Solve for \( a_x \).

\[ a_x = \frac{dv_x}{dt} = \frac{F_x}{m} = \text{constant}. \]

This is a differential equation for \( v_x \), which we solve in effect by separating the variables, thusly:

\[ \int_{v_{ox}}^{v_x} dv_x = \int_{t_o}^{t} a_x dt \]

Integrate both sides

\[ v_x - v_{ox} = a_x (t - t_o). \]

Let \( t_o = 0 \) and solve for \( v_x(t) = v_{ox} + a_x t. \)

Further, \( v_x = \frac{dx}{dt} \), so

\[ \int_{x_o}^{x} dx = \int_{t_o}^{t} (v_{ox} + a_x t) dt. \]

Integrate again and solve for \( x(t) = x_o + v_{ox} t + \frac{1}{2} a_x t^2. \)
2. Examples

a. Inclined plane

\[ \vec{F} = \vec{F}_c + m\vec{g} + \vec{F}_f \]

Decompose normal and parallel to the surface:

\[ x: \quad -mg \sin \theta + F_f = m a_x \]
\[ y: \quad F_c - mg \cos \theta = 0 \]

Substitute \( F_f = \mu F_c \); solve for \( a_x \). Then substitute the constant \( a_x \) into the equations for \( x(t) \) and \( v(t) \).

b. Free fall

\[ a_y = \frac{dv_y}{dt} = -g \]

Separate:

\[ dv_y = -g dt \]

Integrate:

\[ v_y = v_{oy} - gt \]

B. Force as an Explicit Function of Time

\[ \vec{F} = \vec{F}(t) \]

1. Equation of Motion

\[ \frac{dv_x}{dt} = a_x(t) \]
\[ v_x = v_{ox} + \int_0^t a_x(t) dt \]
\[ \frac{dx}{dt} = v_x(t) \]
\[ x = x_o + \int_0^t v_x(t) \, dt \]

We plug in the specified \( a_x(t) \) and integrate twice to obtain \( x(t) \).

2. **Examples**

a. Linear dependence on time

\[ a_x = \frac{d^2 v_x}{dt^2} = b \cdot t \] , where \( b \) is a constant.

\[ v_x = v_{ox} + \int_0^t b \, dt = v_{ox} + \frac{1}{2} b t^2 \]

Similarly, \( \frac{dx}{dt} = v_x \), so that

\[ x = x_o + \int_0^t (v_{ox} + \frac{1}{2} b t^2) \, dt = x_o + v_{ox}t + \frac{1}{6} b t^3. \]

b. Simple harmonic motion

\[ F_x = A \sin(\omega \cdot t) \] , where \( A \) is the amplitude and \( \omega \) is the angular frequency (radians/sec). We integrate twice to obtain \( x(t) \).

\[ v_x = v_{ox} + \int_0^t \frac{A}{m} \sin(\omega \cdot t) \, dt \]

\[ v_x = v_{ox} + \frac{A}{m\omega} \left( -\frac{1}{\omega} \cos(\omega \cdot t) + \frac{\cos(0)}{\omega} \right) \]

\[ v_x = v_{ox} + \frac{A}{m\omega} - \frac{A}{m\omega} \cos(\omega \cdot t) \]

\[ x = x_o + \int_0^t v_x \, dt \]

\[ x = x_o + \int_0^t \left( v_{ox} + \frac{A}{m\omega} - \frac{A}{m\omega} \cos(\omega \cdot t) \right) \, dt \]

\[ x = x_o + \left( v_{ox} + \frac{A}{m\omega} \right) \cdot t - \frac{A}{m\omega^2} \sin(\omega \cdot t) \]

The \( x_o \) and \( v_{ox} \) are initial conditions.

3. **Impulse, a Vector**

a. Definition of impulse

\[ \vec{P} = \int_{t_0}^t \vec{F}(t) \, dt \]
In one dimension, \( P_x = \int_0^t F_x \, dt = \int_0^t m \frac{dv_x}{dt} \, dt = \int_{v_{ox}}^{v_x} m \, dv_x = \int_{p_{ox}}^{p_x} dp_x = \Delta p_x \). The impulse is the momentum imparted to the mass, \( m \), in the time interval \( \Delta t = t - t_o \).

**Definition of impulsive force**: A force that acts such a short time that the mass does not move while the force is acting. The momentum is changed, in effect, instantaneously.

graphically:

The area under the curve gives the magnitude of the impulse, which equals the magnitude of the change in momentum.

b. Example: an object hitting a wall

In an elastic collision, the x-component of the object’s momentum is reversed: \( p_{2x} = -p_{1x} \). If the duration of the collision were \( \Delta t = 0.003 \text{ sec} \), for instance, then the average force exerted on the object by the wall is \( F_x = \frac{p_{2x} - p_{1x}}{\Delta t} \). The equal and opposite force exerted on the wall by the object is \(-F_x\). Care must be taken to keep straight what is exerting which force on what. Often, the exact time-dependence of the force during the collision is not known, though in some cases a force sensor may be used to record the magnitude of the force throughout the impact.

C. Force as a Function of Position

\[ \vec{F} = \vec{F}(\vec{r}) \]

1. **Equation of Motion in One Dimension**

\[ F_x(x) = m \frac{dv_x}{dt} = m \frac{d^2 x}{dt^2} \]

a. Integration

We want to write the equation of motion in terms of \( dx \) because the force is a function of \( x \) rather than of time. Using a chain rule, we obtain

\[ \frac{d^2 x}{dt^2} = \frac{dv_x}{dt} = \frac{dx}{dt} \frac{dv_x}{dx} = v_x \frac{dv_x}{dx} \]

So, we have in the Second “Law”
\[ F_x = m v_x \frac{d v_x}{d x} . \]

But notice that \( \frac{d (v_x^2)}{d x} = 2 v_x \frac{d v_x}{d x} \), so substitute for \( \frac{d v_x}{d x} \) to obtain
\[ F_x = \frac{1}{2} m \frac{d (v_x^2)}{d x} . \]

b. Kinetic energy
In analogy with \( F_x = \frac{d p_x}{d t} \), define the kinetic energy, \( T = \frac{1}{2} m v_x^2 \), such that \( F_x = \frac{d T}{d x} \).

c. Work-energy theorem
In analogy with the definition of impulse, define the work done by a force on the mass, \( m \), as
\[ W = \int_{x_o}^{x} F_x \, dx = T(x) - T(x_o) = \Delta T \]
Keep in mind, this \( W \) is the work done by the force \( F_x \) on the object while the object undergoes a displacement from \( x_o \) to \( x \).

2. Potential Energy Function

a. Potential energy
If the integration limits in the work integral are reversed, then
\[ W = \int_{x}^{x_o} F_x \, dx = G(x) \bigg|_{x}^{x_o} , \] where \( G(x) \) is the antiderivative of \( F_x \). If \( F_x \) is a function of \( x \) only, not of \( t \), or \( v \), or \( v^2 \), etc., then the work integral can be differentiated to obtain
\[ F_x = - \frac{G(x) - G(x_o)}{x - x_o} = - \frac{\Delta G}{\Delta x} \]
Define the potential energy function, \( V \), such that
\[ V(x) - V(x_o) = \int_{x}^{x_o} F_x \, dx , \]
in terms of which \( F_x = - \frac{d V}{d x} \).

b. Total mechanical energy
The right hand side of the definition of \( V(x) \) is also \( - \Delta T \), so
\[ V(x) - V(x_o) = T(x_o) - T(x) \]
\[ V(x) + T(x) = V(x_o) + T(x_o) \]
Define the total mechanical energy as \( E = T + V \). This quantity, \( E \), has the same value at \( x \) as at \( x_o \), even though \( V \) and \( T \) may individually change. If \( T + V \) is indeed constant, then the force \( F_x \) is said to be a conservative force, because the total mechanical energy is conserved. Arguing in a sense backwards, it can be seen that only conservative forces have potential energy functions.
The characteristic property of a conservative force is that the work done by a conservative force depends only on the endpoints of the motion.

c. Motion from the total energy
The equation \( E = T + V \) can be solved for \( v_x(x) \), but what’s really wanted is \( v_x(t) \) and \( x(t) \).

When \( V(x) = E \), \( v_x = 0 \) and the particle turns around. The points \( x_1 \) and \( x_2 \) are called turning points. On the other hand, when \( E - V(x) \) is a maximum, \( v_x \) is maximum, etc. Also, since \( F_x = -\frac{dV}{dx} \), the force can be described qualitatively by looking at the slope of the \( V(x) \) graph.

Quantitatively, the equation of motion is obtained by first solving \( E \) for \( v_x \).

\[
\frac{1}{2} m v_x^2 + V(x) = E
\]

\[
\frac{1}{2} m v_x^2 = E - V(x)
\]

\[
v_x = \frac{dx}{dt} = \sqrt{\frac{2}{m} [E - V(x)]}
\]

Integrate

\[
t = \int_{0}^{x} dt = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m} [E - V(x)]}}
\]

\[
t = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m} [E - V(x)]}}
\]

To go any further, a specific \( V(x) \) is required.

d. Example
Say that \( F_x = -kx \), where \( k \) is a constant. This often called a linear restoring force. Then the potential energy function is

\[
\Delta V(x) = \int_{x}^{x_0} -kx dx = -\frac{1}{2} kx_0^2 + \frac{1}{2} kx^2.
\]

Evidently \( V(x) = \frac{1}{2} kx^2 \). The total mechanical energy is \( E = T + V = \frac{1}{2} m v_x^2 + \frac{1}{2} kx^2 \). With this \( V(x) \),
\[
t = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m} \left( E - \frac{1}{2}kx^2 \right)}} = \sqrt{\frac{m}{2}} \cdot \int_{x_0}^{x} \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}}.
\]

This integral is of the form \( \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{|a|} \), where \( a^2 = E \) and \( u^2 = \frac{1}{2}kx^2 \).

\[
t = \sqrt{\frac{m}{2}} \cdot \int_{u_a}^{u} \frac{2 \, du}{\sqrt{\frac{2}{k}}} = \sqrt{\frac{m}{k}} \left[ \sin^{-1} \frac{u}{|a|} \right]_{u_a}^{u}.
\]

\[
t = \sqrt{\frac{m}{k}} \left[ \sin^{-1} \frac{\sqrt{2k}}{\sqrt{E}} - \sin^{-1} \frac{x_0 \sqrt{2k}}{\sqrt{E}} \right] = \sqrt{\frac{m}{k}} \cdot \sin^{-1} \left( x \sqrt{\frac{k}{2E}} \right) - \Phi.
\]

Solve for \( x(t) \):

\[
x(t) = \sqrt{\frac{2E}{k}} \cdot \sin \left( \sqrt{\frac{k}{m}} \cdot (t + \Phi) \right).
\]

Finally, if desired, the \( v_x(t) = \frac{dx}{dt} = \sqrt{\frac{2E}{m}} \cdot \cos \left( \sqrt{\frac{k}{m}} \cdot (t + \Phi) \right) \). As expected, a linear restoring force leads to simple harmonic motion.

D. Force as a Function of Velocity

An object moving in a fluid experiences a resistive force dependent on the velocity.

1. Equations of Motion—Two Ways to Go

a. First way

\[
F_x(v_x) = ma_x = m \frac{dv_x}{dt}
\]

Separate the variables and integrate

\[
t = \int_{t_0}^{t} \frac{mdv_x}{F_x(v_x)} = \int_{v_{x_0}}^{v_x} \frac{dv_x}{F_x(v_x)}
\]

This expression would be solved for \( v_x(t) \). In turn, \( v_x(t) \) would be integrated to obtain \( x(t) \). A specific form for \( F_x(v_x) \) is required in order to continue.
b. Second way
The alternative approach is to eliminate $t$ to obtain $x(v_x)$. This recalls the procedure used in introductory physics to obtain the four equations of motion (involving $x$, $v$, $a$, and $t$) for cases of constant acceleration. Using the chain rule, the Second “Law” becomes

$$F_x(v_x) = ma_x = m \frac{dv_x}{dt} = m \frac{dv_x}{dx} \frac{dx}{dt} = mv_x \frac{dv_x}{dx}.$$  

Separate the variables and integrate

$$\int dx = \frac{mv_x dv_x}{F_x(v_x)}$$

$$x = x_o + m \int_{v_{ox}}^{v_x} \frac{v_x}{F_x(v_x)} dv_x.$$  

Next, solve for $v_x(x)$. Then, set $v_x(x) = \frac{dx}{dt}$. Then, separate the variables and integrate

$$dt = \frac{dx}{v_x(x)}$$

to obtain $t(x)$. Finally, solve for $x$ in terms of $t$.

In practice, the second method is used when $x(v_x)$ or $v_x(x)$ is all that is desired. The first method is usually shorter in obtaining $v_x(t)$ and $x(t)$.

2. Examples

a. $F_x = -cv_x$, where $c$ is a constant greater than zero.

Follow the steps. . .

$$t = m \int_{v_{ox}}^{v_x} \frac{dv_x}{F_x(v_x)} = m \int_{v_{ox}}^{v_x} \frac{dv_x}{-cv_x} = -\frac{m}{c} \int \frac{dv_x}{v_x}$$

$$t = -\frac{m}{c} \ln \frac{v_x}{v_{ox}}$$

Solve for $v_x$

$$\frac{v_x}{v_{ox}} = e^{-\frac{c}{m} t}$$

$$v_x = v_{ox} e^{-\frac{c}{m} t}$$

In the absence of other forces, the velocity decreases exponentially, with a characteristic time (time constant) of $\tau = \frac{m}{c}$.

Next, integrate to get $x(t)$.

$$\int_{x_o}^{x} dx = v_{ox} \int_{0}^{t} e^{-\frac{c}{m} t} dt = v_{ox} \left[-\frac{m}{c} e^{-\frac{c}{m} t}\right]_0^t$$
\[ x(t) = x_o + \frac{mv_{ax}}{c} \left(1 - e^{-\frac{c}{m}t}\right) \]

Notice that for long times, \( x \to x_o + \frac{mv_{ax}}{c} = \) a constant!

b. \( F_y(v_y) = -mg - cv_y \), where \( mg \) is constant and upward is (+).

Follow those steps. . .

\[ t = m \int_{v_{oy}}^{v_y} \frac{dv_y}{-mg - cv_y} = -m \int_{v_{oy}}^{v_y} \frac{dv_y}{mg + cv_y} \]

\[ t = -m \cdot \ln(mg + cv_y) \bigg|_{v_{oy}}^{v_y} = -m \cdot \ln \left(\frac{mg + cv_y}{mg + cv_{oy}}\right) \]

Take the antilog of both sides

\[ \frac{mg + cv_y}{mg + cv_{oy}} = e^{\frac{c}{m}t} \]

Solve for \( v_y \)

\[ v_y = \frac{1}{c} \left( (mg + cv_{oy}) \cdot e^{\frac{c}{m}t} - mg \right) = \left(\frac{mg}{c} + v_{oy}\right) \cdot e^{\frac{c}{m}t} - \frac{mg}{c} \]

Notice that for long times, \( v_y \to -\frac{mg}{c} = \) a constant called the terminal velocity, \( v_t \). Once terminal velocity is achieved, \( F_y(v_y) = 0 \) and \( v_y \) remains constant thereafter, or until impact.

Finally, \( v_y(t) \) is integrated to obtain

\[ y = y_o - \frac{mg}{c} t + \left(\frac{m^2g}{c^2} + \frac{mv_{oy}}{c} \right) \left( 1 - e^{\frac{c}{m}t} \right) \]

c. \( F_y(v_y) = -mg - \alpha \frac{v_y^3}{|v_y|} \)

This would be too messy to integrate, especially twice. The equation of motion could be solved numerically instead. Firstly, chop time into short intervals, \( \Delta t \), as shown on the time-line below.

Secondly, assume that \( a_y \) is constant during each time step of duration \( \Delta t \). Then the equations for constant acceleration can be used to compute the motion from one time step to the next.
\[
a_y\left(\frac{2}{2}\right) = -g - \frac{\alpha}{m} v_y\left(\frac{1}{2}\right)
\]
\[
y\left(\frac{2}{2}\right) = y\left(\frac{0}{2}\right) + v_y\left(\frac{1}{2}\right) \Delta t
\]
\[
v_y\left(\frac{3}{2}\right) = v_y\left(\frac{1}{2}\right) + a_y\left(\frac{2}{2}\right) \Delta t
\]

Notice that the initial values, \(y(0)\) and \(v_y\left(\frac{1}{2}\right)\) must be given in order to start the cycle of computation. The calculation can be performed by a computer program written in C or Fortran or Basic or some other language, or it can be done in a spreadsheet such as Excel.

E. Harmonic Oscillator

1. Simple Harmonic Oscillator in One Dimension

a. Differential equation of motion

\[
a_x = \frac{F_x}{m}
\]
\[
d^2 x \quad dt^2 = -\frac{k}{m} x
\]

This is a second order ordinary differential equation for \(x(t)\). It can be solved by integration, yielding two constants of integration: \(x_0\) and \(v_{0x}\), as shown in Section IIC. On the other hand, there are only a few functions that are proportional to their own second derivative, such as \(\sin t\), \(\cos t\), or \(e^{q t}\).

\[
\frac{d}{dt} e^{q t} = q e^{q t}
\]
\[
\frac{d^2}{dt^2} e^{q t} = q^2 e^{q t}
\]

b. Proposed solution

Rather than carry out the two integrations, assume a general solution of the form \(x = Ae^{q t} + Be^{-q t}\) and substitute into the differential equation.

\[
\frac{d^2}{dt^2} \left[Ae^{q t} + Be^{-q t}\right] = -\frac{k}{m} \left[Ae^{q t} + Be^{-q t}\right]
\]
\[
Aq^2 e^{q t} + Bq^2 e^{-q t} = -\frac{k}{m} \left[Ae^{q t} + Be^{-q t}\right]
\]
\[
q^2 \left[Ae^{q t} + Be^{-q t}\right] = -\frac{k}{m} \left[Ae^{q t} + Be^{-q t}\right]
\]
Evidently, it must be that \( q^2 = \frac{-k}{m} \), or \( q = i \sqrt{\frac{k}{m}} = i\omega \). The coefficients \( A \) and \( B \) are determined by the initial conditions \( x(0) = x_o \) and \( v_x(0) = v_{ox} \).

\[
Ae^{i\omega t} + Be^{-i\omega t} = x_o
\]
\[
i\omega Ae^{i\omega t} - i\omega Be^{-i\omega t} = v_{ox}
\]

These constitute two equations and two unknowns, which are solved in the usual way.

\[
A + B = x_o
\]
\[
A - B = \frac{v_{ox}}{i\omega}
\]

Add them together \( A = \frac{x_o + \frac{v_{ox}}{i\omega}}{2} \)

Subtract them \( B = \frac{x_o - \frac{v_{ox}}{i\omega}}{2} \)

c. Euler relations

If the sine, cosine, and the exponential functions are expanded in Taylor series, it can be seen that \( e^{i\omega t} = \cos \omega \cdot t + i \sin \omega \cdot t \) and \( e^{-i\omega t} = \cos \omega \cdot t - i \sin \omega \cdot t \). Therefore, the solution obtained in Section IIC is in fact the same as that derived above.

2. Damped Harmonic Oscillator

Suppose the oscillator is immersed in fluid, then perhaps \( F_x = -kx - cv_x \), where \( c \) is the drag coefficient.

a. Equation of motion

\[
ma_x = -kx - cv_x
\]
\[
ma_x + cv_x + kx = 0
\]

In this case, both the first and second derivatives are present. For compactness, let \( \dot{x} = \frac{dx}{dt} \) and \( \ddot{x} = \frac{d^2x}{dt^2} \). Then the equation of motion looks like

\[
m\ddot{x} + c\dot{x} + kx = 0.
\]

b. Solution

Physically, we expect the mass, \( m \), to oscillate like a harmonic oscillator, but with diminishing total mechanical energy because of the viscous resistance term. As before, a solution of the form \( x = Ae^{\sigma t} \) is assumed and substituted into the differential equation.

\[
m\frac{d^2}{dt^2} Ae^{\sigma t} + c \frac{d}{dt} Ae^{\sigma t} + kAe^{\sigma t} = 0
\]
\[
MAq^2e^{\sigma t} + cAe^{\sigma t} + kAe^{\sigma t} = 0
\]
\[ mq^2 + cq + k = 0 \]

This is a quadratic equation for \( q \). The quadratic formula yields two roots.

\[ q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]

Depending of the relative magnitudes of \( c^2 \) and \( 4mk \), there are three cases.

c. Over damping \( c^2 > 4mk \)
The roots are real and negative: Let

\[ \gamma_1 = \frac{c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad \gamma_2 = \frac{c + \sqrt{c^2 - 4mk}}{2m} \]

so that the solution is \( x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t} \). These are both decaying exponentials. Therefore the mass, \( m \), approaches the origin exponentially without oscillation. With the initial velocity set to zero, it looks like:

d. Under damping \( c^2 < 4mk \)
In this case, the roots are complex.

\[ q = -\frac{c}{2m} \pm i \sqrt{\frac{k}{m} - \left( \frac{c}{2m} \right)^2} \]

\[ q = -\gamma \pm i \sqrt{\omega^2 - \gamma^2} = -\gamma \pm i \omega \]

The solution takes the form

\[ x = A_1 e^{-\gamma t + i\omega t} + A_2 e^{-\gamma t - i\omega t} = e^{-\gamma t} [ A_1 e^{i\omega t} + A_2 e^{-i\omega t} ] \]

This function is a harmonic oscillation with an exponentially decaying amplitude.

e. Critically damped
There is one value of \( c^2 = 4mk \), so there is only one double root, \( q = -\frac{c}{2m} = -\gamma \). Now, to complete the solution, two constants are needed, so in this case the assumed solution is not adequate. So, we must return to the original differential equation and solve it by another method.

\[ m\ddot{x} + c\dot{x} + kx = 0 \]

\[ \dot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = \ddot{x} + 2\gamma \dot{x} + \gamma^2 x = 0 \]

Factor this like a polynomial

\[ \left( \frac{d}{dt} + \gamma \right) \left( \frac{d}{dt} + \gamma \right) x = 0 \]

Define a new variable \( u = \left( \frac{d}{dt} + \gamma \right) x \) and solve the following for \( u \).
\[
\left( \frac{d}{dt} + \gamma \right) u = 0
\]
\[
\frac{du}{u} = -\gamma dt
\]
\[
\int \frac{du}{u} = -\int \gamma dt
\]
\[
u = Be^{-\gamma t}
\]
Replace the \(u\),
\[
\left( \frac{d}{dt} + \gamma \right)x = Be^{-\gamma t}
\]
Solve for \(B\),
\[
B = e^{\gamma t} \left( \frac{d}{dt} + \gamma \right)x = \frac{d}{dt} \left( xe^{\gamma t} \right)
\]
Separate the variables again and integrate,
\[
\int_0^t Bdt = \int_A^x d \left( xe^{\gamma t} \right)
\]
\[
Bt = xe^{\gamma t} - A
\]
\[
x = (Bt + A)e^{-\gamma t}
\]
The mass returns to \(x = 0\) more quickly than exponentially, without oscillation.

3. Driven Harmonic Oscillator
An additional force is applied to the damped harmonic oscillator. For instance, we might consider a sinusoidal driving force, \(F_0 e^{i(\omega t + \theta)}\).

a. Equation of motion
\[
m\ddot{x} + c\dot{x} + kx + F_0 e^{i(\omega t + \theta)} = 0
\]
The driving force is independent of \(x\), so we separate the variables.
\[
m\ddot{x} + c\dot{x} + kx = F_0 e^{i(\omega t + \theta)}
\]
This is an inhomogeneous second order ordinary differential equation.

b. Solution
The solution of such a differential equation consists of two parts: the general solution to the homogeneous version, which we have in paragraph 2, plus a particular solution to the inhomogeneous equation. To obtain the particular solution, we use physical insight.

What do we expect? The homogeneous solution dies out with time, so the remaining motion must reflect the time dependence of the driving force. Therefore, we propose that \(x = Ae^{i(\omega t + \delta)}\)
Notice that because the differential equation is second order, the proposed solution has two adjustable constants.
Substitute into the differential equation
\[-mAω^2 e^{i(ωt+δ)} + ciωAe^{i(ωt+δ)} + kA e^{i(ωt+δ)} = F_o e^{i(ωt)}\]

Divide through by the $Ae^{i(ωt+δ)}$
\[-mω^2 + icω + k = \frac{F_o}{A} e^{i(θ-δ)}\]

The real and imaginary parts must be separately equal, so we get two equations with two unknowns.

\[k - mω^2 = \frac{F_o}{A} \cos(θ - δ) = \frac{F_o}{A} \cos φ\]
\[cω = \frac{F_o}{A} \sin(θ - δ) = \frac{F_o}{A} \sin φ\]

The unknowns are $A$ and $φ$ and the simultaneous equations are solved in the usual way. Firstly, divide the second by the first

\[\tan φ = \frac{cω}{k - mω^2}.\]

Secondly, square both equations and add ‘em

\[\left(k - mω^2\right)^2 + c^2ω^2 = \frac{F_o^2}{A^2} \]
\[A^2 = \frac{F_o^2}{\left(k - mω^2\right)^2 + c^2ω^2}\]

In terms of $ω_o^2 = \frac{k}{m}$ and $γ = \frac{c}{2m}$, $A = \frac{F_o}{m} \left[(ω_o^2 - ω^2)^2 + 4γ^2ω^2\right]^{\frac{1}{2}}$.

Thus, the particular solution is $x = \frac{F_o}{m} \left[(ω_o^2 - ω^2)^2 + 4γ^2ω^2\right]^{\frac{1}{2}} e^{i(ωt+δ)}$ where

$φ = θ - δ = \tan^{-1}\left(\frac{2γω}{ω_o^2 - ω^2}\right)$.

c. Interpretation

Notice that $A$ depends on $ω$, as well as $F_o$ and $γ$. It might be asked, at what $ω = ω_r$ is $A$ a maximum? To find that out, set $\frac{dA}{dω} = 0$.

\[-\frac{1}{2m} F_o \left[(ω_o^2 - ω^2)^2 + 4γ^2ω^2\right]^{\frac{3}{2}} \left[2(ω_o^2 - ω^2)(-2ω) + 8γ^2ω\right] = 0\]

The numerator must vanish

\[-4ω(ω_o^2 - ω^2) + 8γ^2ω = 0\]

Rearrange, and rename $ω = ω_r$

$4ω_r(ω_r^2 - ω_o^2 + 2γ^2) = 0$
$ω_r^2 = ω_o^2 - 2γ^2$
This $\omega_r = \sqrt{\omega_o^2 - 2\gamma^2}$ is called the resonant frequency. Plugging $\omega^2 = \omega_r^2 = \omega_o^2 - 2\gamma^2$ back into $A$ gives the maximum amplitude, $A_{\text{max}}$.

$$A_{\text{max}} = \frac{F_o}{m} \left[ (\omega_o^2 - \omega^2 + 2\gamma^2)^2 + 4\gamma^2 \left(\omega_o^2 - 2\gamma^2\right) \right]^{1/2}$$

$$A_{\text{max}} = \frac{F_o}{2m\gamma} \left[ \omega_o^2 - \gamma^2 \right]^{1/2}$$

The phase angle, $\phi$, gives the time difference between the driving force and the resultant motion.

$$\phi = \tan^{-1} \frac{2\gamma\omega}{\omega_o^2 - \omega^2}$$

The mechanical energy of the oscillator is proportional to its amplitude squared. Therefore the driving force transfers maximum energy to the oscillator when the driving frequency is equal to the resonant frequency. In addition, we see that the motion of the oscillator is not in general in phase with the force since $\phi \neq 0$.

d. Quality factor

The resonance peak may be broad or narrow. Let’s take the width to be defined quantitatively as the $\Delta\omega$ between the half-energy points, at $A_{\text{max}}^2/2$.

We’re interested the region $\omega = \omega_o$.

Further, we want weak damping, else there would be no oscillation. Therefore, $\gamma \ll \omega_o$ and $A_{\text{max}} \approx \frac{F_o}{2m\omega_o} \approx \frac{F_o}{c\omega_o}$. Also, $\omega_o^2 - \omega^2 = 2\omega_o(\omega_o - \omega)$ and $\gamma\omega \approx \gamma\omega_o$.

Putting these approximations into $A$ gives

$$A = \frac{F_o}{m \left[ (\omega_o^2 - \omega^2)^2 + 4\gamma^2 \omega^2 \right]^{1/2}} = \frac{\gamma A_{\text{max}}}{\left[ (\omega_o - \omega)^2 + \gamma^2 \right]^{1/2}}.$$
Now, when does $A^2 = \frac{1}{2} A_{\text{max}}^2$? When $(\omega_o - \omega)^2 = \gamma^2$ or when $\omega = \omega_o \pm \gamma$. So $\gamma$ corresponds to $\frac{\Delta \omega}{2}$. A relative measure of the width of the resonance peak is the quality factor, which is defined as $Q = \frac{\omega_d}{\Delta \omega}$, where $\omega_d$ is the frequency of the under damped, but undriven, harmonic oscillator. We saw in paragraph 2 that $\omega_d = \sqrt{\omega_o^2 - \gamma^2}$. If the damping is particularly weak, then $\omega_d \approx \omega_o$ and $Q \approx \frac{\omega_o}{\Delta \omega} = \frac{\omega_o}{2\gamma}$. 
III. Dynamics of a Point in Three Dimensions

A. Extension of the Concepts to Three Dimensions

1. Impulse

a. Newton’s Second “Law”

\[ \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) \]

The force, momentum and velocity are vectors. In three dimensions, we have three component equations, as the vectors are decomposed along the three coordinate axes. E.g.,

\[ F_x = \frac{dp_x}{dt} = \frac{d}{dt}(mv_x); \quad F_y = \frac{dp_y}{dt} = \frac{d}{dt}(mv_y); \quad F_z = \frac{dp_z}{dt} = \frac{d}{dt}(mv_z). \]

b. Force as a function of time; impulse

\[ \vec{P} = \int \vec{F}(t) dt = \int d\vec{p} \]

This equation is solved for \( \vec{v}(t) \), and a second integration yields

\[ \vec{r}(t) = \int \vec{v}(t) dt. \]

Notice that these are vector equations, so they must be decomposed into component equations before the integrals are performed. Also, it is often the case that \( \vec{F}(t) \) is not known, as in collisions. In such cases we rely on the change in momentum.

2. Work-Energy Theorem

a. Kinetic energy

Take the dot product with \( \vec{v} \), from the right on both sides.

\[ \vec{F} \cdot \vec{v} = \left( \frac{d\vec{p}}{dt} \right) \cdot \vec{v} = \frac{d}{dt} \left( \frac{m}{2} \vec{v} \cdot \vec{v} \right), \]

where we have assumed that the mass is constant and have recognized that \( \frac{d}{dt} (\vec{v} \cdot \vec{v}) = 2\vec{v} \cdot \frac{d\vec{v}}{dt} \).

We define the kinetic energy as \( T = \frac{1}{2} mv \cdot \vec{v} \), so that \( \vec{F} \cdot \vec{v} = \frac{d}{dt} T \).

b. Work

\[ \vec{F} \cdot \vec{v} dt = dT \]

Substitute for \( \vec{v} dt = d\vec{r} \)

\[ \vec{F} \cdot d\vec{r} = dT \]

Integrate over the displacement
\[
\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = T_2 - T_1
\]

The left hand side (l.h.s.) is the work done on the particle by the force, as the particle moves from position \( r_1 \) to position \( r_2 \). Of course, to carry out the integral, the vectors must be decomposed into components.

3. Work Integrals

a. Path integrals
In general, the work integral has to be evaluated along the path followed by the particle between the initial and final positions. Further, the applied force is likely not constant. Imagine the path, \( C \), broken up into \( N \) short segments, \( d\vec{r} \). At each segment, the force is \( \vec{F} \). Then the work integral is

\[
\int_{C} \vec{F} \cdot d\vec{r} = \lim_{d\vec{r} \to 0} \sum_{i=1}^{N} \vec{F}_i \cdot d\vec{r}_i.
\]

b. Parameterization of the path
We’d like to reduce the work integral to a one-dimensional integral. In some instances, it is enough to decompose the force and displacement along the coordinate axes. More generally, however, the curve \( C \) may be curved, rather than straight. In that case, we look at the distance traveled along the curved path. Call that distance \( s \).

If \( \vec{F}(\vec{r}) \) and \( \vec{r}(s) \) are known, then
\[
\int_{C} \vec{F} \cdot d\vec{r} = \int \left( \vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int \left( F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_z \frac{dz}{ds} \right) ds
\]

Alternatively, \( \int_{C} \vec{F} \cdot d\vec{r} = \int \left( \vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int F \cos \theta ds \), where \( \theta(s) \) is the angle between the vectors \( \vec{F} \) and \( \vec{r} \) at every point along the path. The choice of parameter will be guided by the symmetry of the path.

c. Example

The work done by gravity on the object as it moves from the top to the bottom of the incline is:

\[
W = \int \vec{F} \cdot d\vec{r} = \int -mg\hat{j} \cdot d\vec{r}
\]
\[ W = \int_{0}^{\sqrt{h^2 + \ell^2}} -mg \cos \phi \, ds = -mg \frac{h}{\sqrt{h^2 + \ell^2}} \sqrt{h^2 + \ell^2} = -mgh \]

d. Example

Consider the work done on a particle moving on a semicircular path in the \( xy \)-plane. The force acting on the particle is \( \vec{F} = -k(\vec{r} - \vec{r}_o) \), where \( \vec{r}_o = a \hat{r} \) and \( a \) is the radius of the semicircle. Let \( s = a(\pi - \alpha) \). Since the motion is confined to a semicircle, we’ll write the work integral first in terms of the arc length, \( s \), and then in terms of the angle \( \alpha \).

\[ \int_{C} \vec{F} \cdot d\vec{r} = \int F \cos \theta \, ds \]

We need to \( F, \theta \), and \( ds \) in terms of the angle, \( \alpha \).

\[ F = k|\vec{r} - \vec{r}_o| = 2ak \sin \frac{\alpha}{2} \] is the magnitude of the force, since by the law of cosines,

\[ \cos \alpha = \frac{r^2 + r_o^2 - |\vec{r} - \vec{r}_o|^2}{2rr_o} = \frac{2a^2 - |\vec{r} - \vec{r}_o|^2}{2a^2}. \]

Solve for

\[ |\vec{r} - \vec{r}_o|^2 = 2a^2(1 - \cos \alpha); \]

\[ |\vec{r} - \vec{r}_o| = 2a \sin \frac{\alpha}{2}. \]

Next, \( \theta = \frac{\pi}{2} - \phi = \frac{\pi}{2} - \left(\frac{\pi - \alpha}{2}\right) = \frac{\alpha}{2} \), by inspection of the diagram. Finally, since the path is clockwise along the semicircle, \( ds = -a \, d\alpha \).

\[ \int_{C} \vec{F} \cdot d\vec{r} = \int_{\alpha}^{\pi} 2ak \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} (-a \, d\alpha) = -2a^2k \int_{\alpha}^{\pi} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \, d\alpha = 2ka^2. \]

4. Potential Energy Functions

a. Conservative force

Condition for exact differential----

A differential \( dV \) is an exact differential if \( \int_{A}^{B} dV = V(B) - V(A) \). If a potential energy function is to be defined for a force, then the quantity \( \vec{F} \cdot d\vec{r} \) must be an exact differential. What requirement on the force insures that this will be so?

Assume that for a given force there is a function such that
\[ \vec{F} = \nabla V = i \frac{\partial V}{\partial x} + j \frac{\partial V}{\partial y} + k \frac{\partial V}{\partial z}. \]

Since these are partial derivatives, \( \frac{\partial F_x}{\partial y} = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial F_y}{\partial x}. \) That is, the order we take the derivatives does not matter. Therefore, we have \( \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \) etc. Now, consider the curl operator

\[ \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = i \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) - j \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) + k \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]

The curl operator (“del cross”) acts on a vector function to produce another vector function, with the usual \( x, y, \) and \( z \) components. The curl is a measure of the extent to which the vector function \( \vec{F}(\vec{r}) \) curls back on itself. It is also associated with rotation.

Referring to the conditions for an exact differential, we see that if a force is a conservative force, then its curl is zero, since \( \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \) etc. This provides a test to determine whether a given force is conservative. If it is conservative, then a potential energy function can be defined for that force. The significance is that the work integral depends only on the end points of the motion, not on the details of the path followed between the initial and final positions.

b. Potential energy

If \( \vec{F} \cdot d\vec{r} \) is an exact differential, then we can write it as \( \vec{F} \cdot d\vec{r} = -dV, \) where \( V(\vec{r}) \) is a scalar function of \( \vec{r}. \) In Cartesian coordinates we have

\[ F_x dx + F_y dy + F_z dz = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \]

We can identify the force components as \( F_x = -\frac{\partial V}{\partial x}, \) etc. since the \( x, y, \) and \( z \)-axes are independent. The \( V(\vec{r}) \) is the potential energy function. It’s a function of position, and the force components are obtained from \( V \) by

\[ \vec{F} = -\frac{\partial V}{\partial x} \hat{i} - \frac{\partial V}{\partial y} \hat{j} - \frac{\partial V}{\partial z} \hat{k} = -\nabla V. \]

The \( \nabla \) operator (called del) is the gradient operator. It may be regarded as the three-dimensional form of the slope. With the minus sign in front, we have defined the potential energy function, or surface, such that the direction of the force will be “down hill.” Notice that the gradient operator acts on a scalar function to produce a vector function.

5. Angular Momentum

a. Return to the Second “Law”
\[ \vec{F} = \frac{d\vec{p}}{dt} \]

Operate from the left with \( \vec{r} \times \)
\[ \vec{r} \times \vec{F}(t) = \vec{r} \times \frac{d\vec{p}}{dt} \]

Now, notice that \( \frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt} \). So, we can write, on the right hand side
\[ \vec{r} \times \vec{F} = \frac{d}{dt}(\vec{r} \times \vec{p}) \]

We define the **angular momentum** of the particle about the origin to be \( \vec{L} = \vec{r} \times \vec{p} \). The value of the angular momentum will depend on the choice of the origin of coordinates. In fact, an angular momentum can be defined about any point, and the vector \( \vec{r} \) points from that point to the particle. Note, too, that the particle need not be moving in a circle or even along a curved path.

The left hand side is called the Moment of Force, or torque. In terms of the angular momentum, we have \( \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \).

**b. Directions**

If the cross product \( \vec{A} \times \vec{B} = \vec{C} \) is written out, it can be seen that the product \( \vec{C} \) is perpendicular to both the vectors \( \vec{A} \) and \( \vec{B} \). Therefore, the direction of the angular momentum vector is perpendicular to both \( \vec{r} \) and \( \vec{p} \). Likewise, the torque is a vector perpendicular to both \( \vec{r} \) and \( \vec{F} \).

**6. Examples**

a. Find the force field of the potential energy function \( V = x^2 + xy + xz \).
\[ \vec{F} = -\nabla V = -i \frac{\partial V}{\partial x} - j \frac{\partial V}{\partial y} - k \frac{\partial V}{\partial z} = -(2x + y + z) \hat{i} - x \hat{j} - x \hat{k} \]

b. Is \( \vec{F} = xy \hat{i} + xz \hat{j} + yz \hat{k} \) a conservative force?
\[ \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix} = (z - x) \hat{i} + (z - x) \hat{k} \neq 0 \]

The force is **not** conservative.
c. Find values of $a$, $b$, and $c$ such that $\mathbf{F} = \left(ax + by^2\right)\hat{i} + cxy\hat{j}$ is a conservative force.

Apply the condition by setting $\nabla \times \mathbf{F} = 0$, then solve the component equations. In this case $\nabla \times \mathbf{F} = (c - 2b)y\hat{k}$. Therefore, $\nabla \times \mathbf{F} = 0$ if $c = 2b$ with any $a$.

B. Separable Forces

In general, the force components may depend on $\vec{r}$, $\vec{v}$, and $t$. This is too hard to deal with analytically. Sometimes, the force is separable. That is, the force components depend only on their respective coordinates; i.e., $F_x = f(x, \dot{x}, t)$ rather than $F_x = f(\vec{r}, \vec{v}, t)$. If $\mathbf{F}$ is separable, then the problem reduces to two (or three) independent one-dimensional problems.

1. Projectile Motion in a Uniform Gravitational Field

a. No air resistance

Newton’s Second “Law”

$$\mathbf{F} = m\ddot{x}$$

Integrate the component equations twice:

$$\ddot{x} = 0 \quad \rightarrow \quad x(t) = \dot{x}_0 t + x_0$$

$$\ddot{y} = 0 \quad \rightarrow \quad y(t) = \dot{y}_0 t + y_0$$

$$\ddot{z} = -g \quad \rightarrow \quad z(t) = -\frac{1}{2}gt^2 + \dot{z}_0 t + z_0$$

The conservation of mechanical energy is

$$\frac{1}{2}m(\ddot{v} \cdot \ddot{v}) + mz = \frac{1}{2}m(\ddot{v}_0 \cdot \ddot{v}_0) + mz_0$$

Consider the trajectory—$z$ as a function of $x$. Basically, we wish to eliminate the variable $t$ from $x(t)$ and $z(t)$.

$$t = \frac{x}{\dot{x}_0},$$

substitute into $z(t)$.
This is the equation for a parabola.

If we similarly eliminate \( t \) from \( x(t) \) and \( y(t) \), we find that \( \frac{y}{x} = \frac{\dot{y}_o}{\dot{x}_o}t = \frac{\dot{y}_o}{\dot{x}_o} = \text{a constant}. \) This result shows that the motion is confined to a vertical plane defined by \( y = \frac{\dot{y}_o}{\dot{x}_o}x \). This is the reason that we treat projectile problems as two dimensional rather than three.

b. linear air resistance

Newton’s Second “Law” says

\[-mg\ddot{k} - c\ddot{v} = m\frac{d^2\ddot{r}}{dt^2}.\]

Decompose

\[\ddot{x} = -\frac{c}{m}\dot{x} \quad \ddot{y} = -\frac{c}{m}\dot{y} \quad \ddot{z} = -g - \frac{c}{m}\dot{z}\]

Just as before, we have three one-dimensional equations of motion. We can make use of previous results:

\[
\dot{x} = \dot{x}_o e^{-\frac{c}{m}t} \\
x = \frac{m}{c} \dot{x}_o \left( 1 - e^{-\frac{c}{m}t} \right) \\
\dot{y} = \dot{y}_o e^{-\frac{c}{m}t} \\
y = \frac{m}{c} \dot{y}_o \left( 1 - e^{-\frac{c}{m}t} \right) \\
\dot{z} = \left( \frac{mg}{c} + \dot{z}_o \right) e^{-\frac{c}{m}t} - \frac{mg}{c} \\
z = \left( \frac{m^2g}{c^2} + \frac{m\dot{z}_o}{c} \right) \left( 1 - e^{-\frac{c}{m}t} \right) - mgt \\
z = \left( \frac{m^2g}{c^2} + \frac{m\dot{z}_o}{c} \right) \left( 1 - e^{-\frac{c}{m}t} \right) - mgt
\]
c. Trajectory with linear air resistance
It’s difficult to visualize the trajectory of a projectile subject to linear air resistance. There are two ways to figure it out, one analytical and one numerical; both are approximate.

i. series expansion
Solve the $x$-equation for $t = -\frac{m}{c} \ln \left( 1 - \frac{cx}{mx_o} \right)$ and substitute into the $z$-equation.

Expand the $\ln$ function in a series

$$\ln \left( 1 - \frac{cx}{mx_o} \right) = -\frac{cx}{mx_o} - \frac{1}{2} \left( \frac{cx}{mx_o} \right)^2 + \frac{1}{3} \left( \frac{cx}{mx_o} \right)^3 - \frac{1}{4} \left( \frac{cx}{mx_o} \right)^4 + \cdots$$

Whence $z(x) = \left( \frac{mg}{c\dot{x_o}} + \frac{\dot{z_o}}{x_o} \right) x - \frac{mg}{c\dot{x_o}} x - \frac{1}{2} \frac{g}{x_o^2} x^2 - \frac{1}{3} \frac{cg}{3mx_o^3} x^3 - \cdots$. Collect the powers of $x,

$$z(x) = \frac{\dot{z_o}}{x_o} x - \frac{1}{2} \frac{g}{x_o^2} x^2 - \frac{1}{3} \frac{cg}{3mx_o^3} x^3 - \cdots$$

Now, the first two terms are the parabolic trajectory of a particle experiencing no air resistance. The added terms give corrections to that trajectory due to air resistance.

ii. numerical simulation
The alternative is to perform a numerical solution to the equations of motion, just as was done in section II. D. 2. for the one-dimensional case. Now there will be six equations, since we have two dimensions.

$$\dot{x} \left( \frac{2}{2} \right) = -cx \left( \frac{1}{2} \right)$$
$$\dot{z} \left( \frac{2}{2} \right) = -g - cz \left( \frac{1}{2} \right)$$
$$x \left( \frac{2}{2} \right) = x \left( \frac{0}{2} \right) + \dot{x} \left( \frac{1}{2} \right) \Delta t$$
$$z \left( \frac{2}{2} \right) = z \left( \frac{0}{2} \right) + \dot{z} \left( \frac{1}{2} \right) \Delta t$$
$$\dot{x} \left( \frac{3}{2} \right) = \dot{x} \left( \frac{1}{2} \right) + \dot{x} \left( \frac{2}{2} \right) \Delta t$$
$$\dot{z} \left( \frac{3}{2} \right) = \dot{z} \left( \frac{1}{2} \right) + \dot{z} \left( \frac{2}{2} \right) \Delta t$$

We may imagine further elaborations, such as a drag coefficient that depends on altitude, etc.
2. Harmonic Oscillator

a. isotropic harmonic oscillator

In the Second “Law”, \( m\ddot{a} = -k\vec{r} \).

Separate the variables.

\[
\begin{align*}
    m\ddot{x} &= -kx \\
    m\ddot{y} &= -ky \\
    m\ddot{z} &= -kz
\end{align*}
\]

We have three one-dimensional problems, which we solved already.

\[
\begin{align*}
    x(t) &= A_1 \sin \omega \cdot t + A_2 \cos \omega \cdot t \\
    x(t) &= A_1 \sin \omega \cdot t + A_2 \cos \omega \cdot t \\
    x(t) &= A_1 \sin \omega \cdot t + A_2 \cos \omega \cdot t
\end{align*}
\]

All have the same angular frequency, \( \omega = \sqrt{\frac{k}{m}} \).

To obtain a trajectory, we eliminate \( t \) to get, for instance,

Step one:

\[
\begin{align*}
    B_1 x &= B_1 A_1 \sin \omega \cdot t + B_1 A_2 \cos \omega \cdot t \\
    - A_1 y &= -A_1 B_1 \sin \omega \cdot t - A_1 B_2 \cos \omega \cdot t
\end{align*}
\]

Solve for

\[
\cos \omega \cdot t = \frac{B_1 x - A_1 y}{B_1 A_2 - A_1 B_2}
\]

Step two:

\[
\begin{align*}
    B_2 x &= B_2 A_1 \sin \omega \cdot t + B_2 A_2 \cos \omega \cdot t \\
    - A_2 y &= -A_2 B_1 \sin \omega \cdot t - A_2 B_2 \cos \omega \cdot t
\end{align*}
\]

Solve for

\[
\sin \omega \cdot t = \frac{B_2 x - A_2 y}{B_2 A_1 - A_2 B_1}
\]

Step three: Substitute these into \( z(t) \).

\[
\begin{align*}
    z(x, y) &= C_1 \left( \frac{B_2 x - A_2 y}{B_2 A_1 - A_2 B_1} \right) + C_2 \left( \frac{B_1 x - A_1 y}{B_1 A_2 - A_1 B_2} \right) \\
    z(x, y) &= x \left( \frac{C_1 B_2 - C_2 B_1}{B_2 A_1 - A_2 B_1} \right) + y \left( \frac{C_2 A_1 - C_1 A_2}{B_2 A_1 - A_2 B_1} \right)
\end{align*}
\]
The items in the brackets are constants, determined by initial conditions. So, \( z(x,y) \) is of the form of an equation for a plane. That is, the motion of the three dimensional harmonic oscillator is confined to a two dimensional plane.

b. Non-isotropic harmonic oscillator

More generally, the force constants may not be the same in the three directions.

\[
\vec{F}(\vec{r}) = -k_x \hat{x} - k_y \hat{y} - k_z \hat{z} = -\vec{k} \cdot \vec{r}.
\]

The separated equations of motion are

\[
m\ddot{x} = -k_x x \\
m\ddot{y} = -k_y y \\
m\ddot{z} = -k_z z.
\]

These have solutions of the same form, but different constants.

\[
x(t) = A_x \sin \omega_x t + A_2 \cos \omega_x t \\
y(t) = B_y \sin \omega_y t + B_2 \cos \omega_y t \\
z(t) = C_z \sin \omega_z t + C_2 \cos \omega_z t
\]

where \( \omega_x = \sqrt{\frac{k_x}{m}} \), \( \omega_y = \sqrt{\frac{k_y}{m}} \), and \( \omega_z = \sqrt{\frac{k_z}{m}} \).

Special case: if \( \frac{\omega_x}{n_x} = \frac{\omega_y}{n_y} = \frac{\omega_z}{n_z} \), where \( n_x, n_y, \) and \( n_z \) are integers, then the frequencies are said to be commensurable, and the trajectory of the oscillator is closed. These closed trajectories are the famous Lissajou Figures.

c. Energy

The potential energy function is \( V(x, y, z) = \frac{1}{2} \left(k_x x^2 + k_y y^2 + k_z z^2\right) \) and the kinetic energy is \( T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) \). The total mechanical energy is conserved \( E = T + V = \) a constant.
C. Constrained Motion of a Particle

If a moving particle is restricted to a definite curve or surface, its motion is said to be constrained. A constraint may be complete or one-sided, fixed or moving. We will consider only fixed constraints. This is not really a new issue, just an alternative way of thinking about the forces acting on a particle.

1. Smooth Constraints

a. Energy considerations

Let’s suppose that the forces acting on a particle are divided into two classes—call them the applied forces and the constraining forces. The constraining forces are those that confine the particle’s motion to a specific path or surface. In the Second “Law”

\[ \vec{F} + \vec{R} = m \frac{d\vec{v}}{dt} \]

Take the dot product with \( \vec{v} \) on both sides of the equation

\[ \vec{F} \cdot \vec{v} + \vec{R} \cdot \vec{v} = m \left( \frac{d\vec{v}}{dt} \right) \cdot \vec{v}. \]

Now, if \( \vec{R} \perp \vec{v} \) everywhere along the path, then \( \vec{R} \cdot \vec{v} = 0 \). We have remaining

\[ \vec{F} \cdot \vec{v} = m \left( \frac{d\vec{v}}{dt} \right) \cdot \vec{v} = \frac{d}{dt} \left( \frac{1}{2} m \vec{v} \cdot \vec{v} \right). \]

If the force, \( \vec{F} \), is conservative, we can integrate to obtain \( \frac{1}{2} m v^2 + V(x, y, z) = E = \text{constant}, \) just as before. That is, the constraining forces do no work. Such constraints are called smooth.

b. Example—a classic

A particle begins on top of a smooth sphere of radius \( a \). If it begins to slide, where will it begin to leave the sphere’s surface?

The forces acting are \( \vec{F} = -mg\hat{k} \) and \( \vec{R} \), the contact or normal force. In the Second “Law”

\[ \vec{R} - mg\hat{k} = m \frac{d\vec{v}}{dt}. \]

The key is to recognize that when the particle leaves the sphere, the contact force vanishes. So we solve for \( \vec{R} \) and set it equal to zero.

The radial components are \( -mg \cos \theta + R = -m \frac{v^2}{a} \). Solve for \( R \).

\[ R = -m \frac{v^2}{a} + mg \cos \theta \]
The next step is to parameterize \( v \) in terms of \( \theta \), or both \( v \) and \( \theta \) in terms of \( z \). From the conservation of energy we get \( \frac{1}{2}mv^2 + mgz = E = mga \), whence \( v^2 = 2g(a - z) \). Also, \( \cos \theta = \frac{z}{a} \). Plugging these into the equation for \( R \) yields

\[
R = -\frac{m}{a} 2g(a - z) + mg \frac{z}{a} = 0
\]

Solve for \( z \)

\[
-2g + 2g \frac{z}{a} + g \frac{z}{a} = 0
\]

\[
z = \frac{2}{3}a
\]

2. **Motion on a Curve**

A particle is constrained to move along a specified curve, \( C \). Its displacement is \( \vec{r}(x, y, z) \). In terms of \( s \), the distance traveled along \( C, x = x(s), y = y(s), \) and \( z = z(s) \). There are two approaches, both starting from the total energy, \( E \).

a. **Energy**

\( E = \frac{1}{2}m\dot{s}^2 + V(s) = \text{constant.} \) This is solved for \( s \) by integration:

\[
t = \int_0^s \frac{mds}{2\sqrt{E - V(s)}}.
\]

b. **Tangential force**

Alternatively, differentiate the total energy equation

\[
m\ddot{s} + \frac{dV}{ds} = 0,
\]

where the quantity \( \frac{dV}{ds} = -F_s \) is the negative of the tangential component of the external force, \( \vec{F} \), that is tangential to the curve, \( C \). As an example, revisit the simple pendulum.
\[ V(s) = mgz = mg\ell (1 - \cos \theta) = mg\ell - mg\ell \cos \left( \frac{s}{\ell} \right) \]

\[-\frac{dV}{ds} = -mg \sin \theta = -mg \sin \left( \frac{s}{\ell} \right) \]

So we have \( m\ddot{s} + mg \sin \left( \frac{s}{\ell} \right) = 0 \). We would integrate this twice to obtain \( s(t) \). If \( s \ll \ell \), then of course we have simple harmonic motion. If not, the motion is not simple harmonic. Instead,

\[ \ddot{s} = -g \sin \left( \frac{s}{\ell} \right); \]

\[ \ddot{s} = -g \left\{ \frac{s}{\ell} - \frac{1}{3!} \left( \frac{s}{\ell} \right)^3 + \frac{1}{5!} \left( \frac{s}{\ell} \right)^5 - \cdots \right\}. \]

Integrate \emph{that} twice in your spare time.
IV. Accelerated Reference Frames

A. Galilean Transformation

1. Transformation Equations

The motion of a particle is observed relative to O and to O'. The Galilean Transformation of coordinates is:

\[
\vec{r} = \vec{r}' + \vec{r}_o
\]

\[
\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{r}_o}{dt} = \vec{v}' + \vec{v}_o
\]

\[
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}'}{dt} + \frac{d\vec{v}_o}{dt} = \vec{a}' + \vec{a}_o
\]

The relative displacement, velocity, and acceleration are measured with respect to O.

2. Translating Reference Frames

Consider a mass hanging in static equilibrium inside a railcar. That is, if the O' frame is inside the railcar, then \( \vec{a}' = 0 \). The railcar is accelerating along the straight, smooth track with acceleration \( \vec{a}_o \).

In the O-frame, the free body diagram looks like this:

In the Second “Law”,

\[ \vec{T} + \vec{W} = m\vec{a} = m\vec{a}_o. \]

The component equations are

\[ T \sin \theta = ma = ma_o \] and \[ T \cos \theta - mg = 0. \]

As observed in the O'-frame, the Second “Law” says

\[ \vec{T}' + \vec{W}' = m\vec{a}' = 0. \]

The component equations are

\[ T' \sin \theta - W' \sin \theta = 0 \] and \[ T' \cos \theta - W' \cos \theta = 0, \]

where naturally one would say that the weight of the hanging mass is \( W' = mg' \) and that \( \vec{W}' \) makes an angle \( \theta \) with the vertical.
Suppose that somehow the external observer (in the $O$-frame) could inform the observer inside the railcar that the force of gravity in fact pointed straight down, and that the “true” weight of the mass is $W = mg$. Then the component equations in the $O'$-frame would look like this:

$$T' \sin \theta - f_x = 0 \quad \text{and} \quad T' \cos \theta - mg = 0.$$  

Of course $T' = T$, so we can identify the $f_x = ma$, This $f_x$ is called a fictitious or an inertial force, since it arises from the relative acceleration of the reference frames, not from interaction among physical bodies.

**B. Rotating Reference Frames**

1. **Equations of Motion**

   a. Rotating axes

   Consider two coordinate frames, one rotating with respect to the other. The angular velocity of the rotating frame ($O'$) is $\omega$. The direction is given by the right-hand-rule. In what follows, we will have $\frac{d'}{dt}$ and $\frac{d}{dt}$, derivatives taken in the $O'$- and $O$-frames, respectively.

   b. Rotating vectors

   Now, let there be some physical vector, $\vec{B}$. Its components in each of the two frames are:

   $$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \quad \text{and} \quad \vec{B} = B'_x \hat{i}' + B'_y \hat{j}' + B'_z \hat{k}'. $$

   The $O$-frame is fixed; the $O'$-frame rotates. We want to examine how the components of $\vec{B}$ change with time. For the moment, assume that $\vec{B}$ is fixed in the $O'$-frame, so that $\frac{d'\vec{B}}{dt} = 0$.

   Geometrically, we see that in the $O$-frame

   $$\Delta \vec{B} = (\omega \cdot \Delta t) \cdot \vec{B} \sin \theta,$$

   where $\theta$ is the angle between $\vec{B}$ and $\vec{\omega}$.

   $$\frac{\Delta \vec{B}}{\Delta t} = \omega \vec{B} \sin \theta$$

   Take the limit as $\Delta t \to 0$,

   $$\frac{d \vec{B}}{dt} = \omega \vec{B} \sin \theta$$

   From the diagram, the direction is perpendicular to both $\vec{B}$ and $\vec{\omega}$. That is,
\[
\frac{d\vec{B}}{dt} = \dot{\omega} \times \vec{B}.
\]

Now we turn to the more general time derivative of \( \vec{B} \).

\[
\frac{d\vec{B}}{dt} = \frac{d}{dt} \left( B_x \dot{i} + B_y \dot{j} + B_z \dot{k} \right)
\]

\[
= \frac{dB_x}{dt} \dot{i} + \frac{dB_y}{dt} \dot{j} + \frac{dB_z}{dt} \dot{k} + \omega \times (\dot{i} \times \vec{B}) + \dot{\omega} \times \vec{B}
\]

\[
= \frac{d\vec{B}}{dt} + \omega \times \vec{B}
\]

c. Motion variables

If, at the moment, the origins of the two reference frames coincide, then \( \vec{r} = \vec{r}' \). To obtain the velocity of the particle, we take the time derivative of \( \vec{r} \).

\[
\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \dot{\omega} \times \vec{r}'
\]

Similarly, the acceleration is

\[
\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \dot{\omega} \times \vec{r}'
\]

If translational motion or acceleration of the \( O' \)-frame is allowed, then

\[
\frac{d\vec{v}}{dt} = \frac{d\vec{v}'}{dt} + \dot{\omega} \times \vec{r}'
\]

From the point of view of an observer in the rotating \( O' \)-frame, we solve the last equation for \( \vec{v} \), which goes into Newton’s Second “Law” as written in the \( O' \)-frame:

\[
\ddot{\vec{r}}' = m\ddot{\vec{r}}.
\]

Each term has a name:

- \( m\ddot{r} \): real physical net force, such as gravity
- \(-m\ddot{\omega} \times \vec{r} \): inertial “force”
- \(-2m\ddot{\omega} \times \vec{r}' \): Coriolis effect
- \(-m\ddot{\omega} \times (\dot{\omega} \times \vec{r}') \): centrifugal “force”
- \(-m\ddot{\omega} \times \vec{r} \): transverse “force”
The latter four “forces” are called fictitious forces, as they arise from the acceleration of the reference frame, not from a physical interaction among particles. However, their effects are not necessarily readily distinguishable from real physical forces in the $O'$-frame.

d. Example

turn table, merry-go-round, etc.

Let $\ddot{\omega} = \omega \cdot \hat{k}' = \text{constant}; \quad \ddot{r}' = x'\hat{i}' ; \quad \dddot{r}' = v_o\dddot{i}' = \text{constant}; \quad \text{and} \quad \dddot{r}' = 0$.

As experienced in the rotating reference frame, $\vec{F}'$ is the net force...

$$\vec{F}' = m\ddot{r}' = 0$$
$$\vec{F} - m\ddot{\omega} \times \dot{r}' - m\ddot{\omega} \times (\vec{\omega} \times \ddot{r}') - m\dddot{\omega} \times \dot{r}' = 0$$
$$\vec{F} - 2m\dddot{\omega} \times \dot{r}' - m\dddot{\omega} \times (\vec{\omega} \times \ddot{r}') = 0$$
$$\vec{F} - 2m\omega \cdot \dddot{k}' \times \dddot{v}_o\dddot{i}' - m\omega \cdot \dddot{k}' \times (\vec{\omega} \cdot \dddot{k}' \times \dddot{r}') = 0$$
$$\vec{F} - 2m\omega \cdot v_o\dddot{j}' \times \dddot{r}' + m\omega^2 x' \dddot{i}' = 0$$

The applied force, $\vec{F}$, is the force (such as friction) necessary to maintain the constant $\dddot{r}'$.

At the same instant, an observer in the $O$-frame would write:

$$\dddot{r} = \dddot{r}' = x'\hat{i}'$$
$$\dddot{r} = \dddot{r}' + \dddot{\omega} \times \dddot{r}' = v_o\dddot{i}' + \dddot{\omega} \times \dddot{r}'$$
$$\dddot{r} = \dddot{r}' + \dddot{\omega} \times \dddot{r}' + 2\dddot{\omega} \times \dddot{r}' + \dddot{\omega} \times (\dddot{\omega} \times \dddot{r}') = 0 + 2\dddot{\omega} \cdot v_o\dddot{j}' - \omega^2 x' \dddot{i}' = \frac{\vec{F}}{m}$$

These deceptively simple expressions arise because we have taken a “snapshot” of the turn table at the instant when the $xyz$-axes coincide with the $x'y'z'$-axes. We have not obtained $\vec{F}(t)$ as yet.

e. rotating velocity vector

By way edging into the full blown treatment of rotating reference frames, consider a turn table again, rotating with constant angular velocity. An insect walks with constant speed outward from the center along the $\dddot{i}'$-axis. That is, $\dddot{r}' = v_o\dddot{i}' = \text{constant}$. What are the velocity components relative to the non-rotating axes? Well, $\dddot{r} = \dddot{r}' + \dddot{\omega} \times \dddot{r}' = v_o\dddot{j}' + \dddot{\omega} \times \dddot{x}' \dddot{i}'$. But this gives the components relative to the rotating axes. We need relations between the unit vectors of the two reference frames.

$$\dddot{i}' = \dddot{i} \cos \theta + \dddot{j} \sin \theta$$
$$\dddot{j}' = \dddot{i} (-\sin \theta) + \dddot{j} \cos \theta$$
$$\dddot{k} = \dddot{k}'$$

Substituting for the unit vectors, we obtain

$$\dddot{r} = v_o (\dddot{i} \cos \theta + \dddot{j} \sin \theta) + \dddot{\omega} \cdot x' (\dddot{i} \sin \theta + \dddot{j} \cos \theta).$$
What are \( x' \) and \( \theta' \)? \( x' = v_0 t \) and \( \theta = \omega \cdot t \). So, we have at last
\[
\dot{r} = i v_0 (\cos \omega \cdot t - \omega \cdot t \sin \omega \cdot t) + j v_0 (\sin \omega \cdot t - \omega \cdot t \cos \omega \cdot t).
\]

2. Rotating Earth

a. Statics

Consider a plumb bob hanging at rest, and let the origin of our coordinate system be at the bob, so \( \ddot{r}' = \ddot{r} = 0 \). The "real" physical forces on the bob are gravity and the tension in the cord. We have also \( \dot{\omega} = 0 \) and \( \ddot{r}' = 0 \).

As seen in the rotating frame: \( \dddot{T}' + \dddot{W}' = 0 \). Since we use plumb bobs to determine the vertical direction, we'd say that \( T' - mg' = 0 \). However, since the Earth is rotating, \( \dddot{W}' \) does not point toward the center of the Earth. As seen in an inertial frame, Newton's 2nd "Law" appears as \( \dddot{T} + \dddot{W} = m \dddot{a}_o \), where \( \dddot{a}_o = \frac{v^2}{\rho} = \rho \omega^2 \) is the centripetal acceleration, directed toward the Earth's rotational axis. Now, \( \dddot{T} = \dddot{a}_o \), so we can write \( -\dddot{W}' = -\dddot{W} + m \dddot{a}_o = 0 \), or \( \dddot{W}' = \dddot{W} - m \dddot{a}_o \). The deviation from the true vertical (pointing toward the center of the Earth), is a function of the latitude, \( \lambda \), since
\[
\begin{align*}
\sin e & = \frac{\sin \lambda}{g} \\
\sin e \rho & = \frac{\sin \lambda}{g'} \\
\sin e \rho \omega^2 & = \frac{\sin \lambda}{g'} \\
R \cos \lambda \omega^2 & = \frac{\sin \lambda}{g'} \\
\sin e & = \frac{R \omega^2}{2g'} \sin(2\lambda)
\end{align*}
\]

Our scales actually measure \( W' = mg' \). However, since the Earth rotates slowly, \( g' \approx g \). The same would not be true on a planet that rotates much faster than the Earth. See for instance the science fiction novel *A Mission of Gravity* by Hal Clement.

b. Dynamics
A frame fixed to the Earth’s surface is the $O’$-frame. In that frame, we observe $\ddot{r}', \dot{r}'$ and $\ddot{r}'$. In the rotating frame, Newton’s 2nd “Law” looks like $\vec{F}' = m\ddot{r}'$. As related to an inertial reference frame, $\ddot{r}' = \ddot{r} - \ddot{a}_o - 2\ddot{a} \times \dot{r}' - \omega \times (\omega \times \dot{r}') - \dot{\omega} \times \dot{r}'$. On the Earth, $a_o = R \cos \lambda \omega^2$, $\omega = \frac{2\pi}{day} = 7.3 \times 10^{-5} \frac{rad}{sec}$, $\ddot{a}_o = 0$, and the product $\ddot{a} \times (\ddot{a} \times \dot{r}')$ is very small. For a projectile without air resistance, $m\ddot{r}' = \vec{W}$ only. The equation of motion reduces to $m\ddot{r}' = \vec{W} - 2m\omega \times \dot{r}'$. We may further neglect $a_o \approx \omega^2$ in this case. [Since $\omega$ is small, $\dot{\omega} = -mg\hat{k}'$.]

Let $\hat{i}'$ be east, $\hat{j}'$ be north, and $\hat{k}'$ be the local vertical. Then our vectors have the following components:

$$\vec{W}' = -mg\hat{k}'$$
$$\ddot{r} = \omega_x\hat{i}' + \omega_y\hat{j}' + \omega_z\hat{k}' = \omega \cos \lambda \hat{j}' + \omega \sin \lambda \hat{k}'$$.

The Coriolis term is

$$\ddot{r} \times \dot{r}' = \begin{vmatrix} \ddot{i}' & \ddot{j}' & \ddot{k}' \\ \omega_x & \omega_y & \omega_z \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix} = \dot{i}'(\omega \cdot \dot{z}' \cos \lambda - \omega \cdot \dot{y}') + \dot{j}'(\omega \cdot \dot{x}' \sin \lambda) + \dot{k}'(-\omega \cdot \dot{x}' \cos \lambda).$$

Therefore, the components of $\ddot{r}'$ are

$$\dot{x}' = -2\omega \cdot (\dot{z}' \cos \lambda - \dot{y}' \sin \lambda)$$
$$\dot{y}' = -2\omega \cdot (\dot{x}' \sin \lambda)$$
$$\dot{z}' = -g' + 2\omega \cdot (\dot{x}' \cos \lambda)$$

[Recall that $g' = g$ in this case.]

These are coupled equations; we want to de-couple them. Firstly, integrate each one with respect to time. . .

$$\dot{x}' + 2\omega \cdot (\dot{z}' \cos \lambda - \dot{y}' \sin \lambda) + x'_o$$
$$\dot{y}' + 2\omega \cdot (\dot{x}' \sin \lambda) + y'_o$$
$$\dot{z}' - g't\ + 2\omega \cdot (\dot{x}' \cos \lambda) + z'_o$$

Substitute $\dot{y}'$ and $\dot{z}'$ into $\dot{x}'$.

$$\dot{x}' = -2\omega \cdot [(\dot{y}' + 2\omega \cdot \dot{x}' \cos \lambda + \dot{z}'_o) \cos \lambda - (\dot{y}' + 2\omega \cdot \dot{x}' \sin \lambda + \dot{y}'_o) \sin \lambda]$$

$$\dot{y}' = (2\omega \cdot g't - 4\omega^2 \dot{x}' \cos \lambda - 2\omega \cdot \dot{z}'_o) \cos \lambda + (4\omega^2 \dot{x}' \sin \lambda + 2\omega \cdot \dot{y}'_o) \sin \lambda$$

We neglect terms in $\omega^2$ as being much smaller than terms in $\omega$.

$$\dot{x}' = (2\omega \cdot g't - 2\omega \cdot \dot{z}'_o) \cos \lambda + 2\omega \cdot \dot{y}'_o \sin \lambda$$
$$\dot{y}' = 2\omega \cdot g't \cos \lambda - 2\omega \cdot (\dot{z}'_o \cos \lambda - \dot{y}'_o \sin \lambda)$$
Integrate.

\[ \dot{x}' = \omega \cdot g \cdot t^2 \cos \lambda - 2\omega \cdot t(\ddot{z}_o \cos \lambda - \dot{y}_o \sin \lambda) + \dot{x}_o' \]

Integrate again.

\[ x' = \frac{1}{3} \omega \cdot g \cdot t^3 \cos \lambda - \omega \cdot t^2(\ddot{z}_o \cos \lambda - \dot{y}_o \sin \lambda) + \dot{x}_o' t + x_o' \]

Penultimately, plug this into \( \dot{y}' \) and \( \dot{z}' \).

\[ \begin{align*}
\dot{y}' &= -2\omega \cdot \{x'\} \sin \lambda + \dot{y}_o' \\
\dot{z}' &= -g' t + 2\omega \cdot \{x'\} \cos \lambda + \dot{z}_o'
\end{align*} \]

Integrate each of these, afterward dropping once again terms in \( \omega^2 \).

\[ \begin{align*}
y' &= \dot{y}_o' t - \omega \cdot \dot{x}_o' t^2 \sin \lambda + y_o' \\
z' &= -\frac{1}{2} g' t^2 + \dot{z}_o' t + \omega \cdot \dot{x}_o' t^2 \cos \lambda + z_o'
\end{align*} \]

c. Projectile

Example: an object dropped from rest. \( x_o' = y_o' = z_o' = 0 \) and \( \dot{x}_o' = \dot{y}_o' = \dot{z}_o' = 0 \). Then,

\[ \begin{align*}
x' &= \frac{1}{3} \omega \cdot g' \cdot t^3 \cos \lambda \\
y' &= 0 \\
z' &= -\frac{1}{2} g' t^2
\end{align*} \]

We see that \( x' \neq 0 \); the object drifts to the east as it falls. Suppose \( z' = -h \). If we can pretend that the object is falling nearly straight downward, then the fall-time is \( t^2 = \frac{2h}{g} \) and

\[ x' = \frac{1}{3} \omega \cdot g' \cos \lambda \left( \frac{8h^3}{g'^3} \right)^{\frac{1}{2}}. \]

Example: an object projected horizontally at high velocity. \( x_o' = v_o \) and \( \dot{y}_o' = \dot{z}_o' = 0 \)

\[ \begin{align*}
x' &= v_o t + \frac{1}{3} \omega \cdot g' \cdot t^3 \cos \lambda \\
y' &= -\omega \cdot v_o \cdot t^2 \sin \lambda
\end{align*} \]

Remember, \( \hat{i}' \) is east and \( \hat{j}' \) is horizontal, parallel to the Earth’s surface. So the projectile drifts to the right \((-\hat{j}')\) at a rate proportional to \( v_o \). If \( R \) is the horizontal range, then \( t = \frac{R}{v_o} \) is the time of flight and the total drift in the \( \hat{j}' \)-direction is \(-\omega \cdot \frac{R^2}{v_o} \sin \lambda \).

Of course, the horizontal range is determined in the first place by
how long it takes for the projectile to fall from its initial elevation to its final elevation. If the change in elevation is given, then the range is approximately \( R = v_o \sqrt{\frac{2\Delta z'}{g}} \).

However, if \( h \) (or \( \Delta z' \)) is large, then we can no longer use the approximate fall-time. If air resistance is added as well, then it’s time for the numerical solution.

d. Foucault pendulum

We are not sensibly aware of sitting or standing on a rotating surface. A definitive physical demonstration that the Earth rotates is the Foucault Pendulum.

Assume that the Earth rotates with a constant angular velocity, \( \omega \). As viewed in the rotating frame, the equation of motion for the pendulum bob is

\[
\ddot{r}' = \ddot{\bar{T}}' + \dddot{\bar{W}}' - 2m\ddot{\omega} \times \dot{\bar{r}}'.
\]

Decompose

\[
m\ddot{x}' = -\frac{\dot{x}'}{\ell} T' - 2m\omega \cdot (\dot{z}' \cos \lambda - \dot{y}' \sin \lambda)
\]

\[
m\ddot{y}' = -\frac{\dot{y}'}{\ell} T' - 2m\omega \cdot \dot{x}' \sin \lambda
\]

\[
m\ddot{z}' = T'_{z'} - mg' + 2m\ddot{\omega} \cdot \dot{x}' \cos \lambda
\]

As usual, we consider small oscillations, in which case \( T' = \bar{W}' = mg' \) and \( \dot{z}' \ll \dot{y}' \). Then

\[
\dot{x}' = -\frac{g}{\ell} x' + 2\omega \cdot \dot{y}' \sin \lambda \quad \text{and}
\]

\[
\dot{y}' = -\frac{g}{\ell} y' - 2\omega \cdot \dot{x}' \sin \lambda.
\]

The pendulum bob experiences a transverse force which causes the plane of its swing to precess about the \( \hat{k}' \)-axis, and at a rate proportional to \( \omega \) and to the sine of the latitude, \( \lambda \). The fact that just exactly such a precession is observed serves to demonstrate that the Earth rotates.
V. Potpourri

A. Systems of Particles

1. \(N\)-particles
Consider a system comprised of \(N\) particles, each with its own mass, \(m_i\), its own position, \(\vec{r}_i\), and velocity, \(\vec{v}_i\), and acceleration, \(\vec{a}_i\). The particles exert forces on one another and may be subject to forces external to the system as well.

a. Center of Mass
\[
\vec{R} = \frac{\sum m_i \vec{r}_i}{M},
\]
where \(M\) is the total mass of the system. The velocity and acceleration of the center of mass is obtained by the usual time-derivatives.

\[
\vec{V} = \frac{\sum m_i \vec{v}_i}{M}, \text{ and } \vec{A} = \frac{\sum m_i \vec{a}_i}{M}.
\]
Keep in mind that there need be no particle located at the center of mass.

b. Forces
Newton’s Second “Law” applies to each particle: \(\sum_{i=1}^{N} \vec{F}_{ik} = m_i \ddot{r}_i\). Of course, the \(i = k\) term is omitted from the summation. \(\vec{F}_i\) is the total external force acting on \(m_i\). \(\vec{F}_{ik}\) is the force exerted by the \(k\)th particle on the \(i\)th particle.

The Second “Law” applies to the system as a whole, also. The total net force on the system is

\[
\sum_{i} \vec{F}_i + \sum_{i} \sum_{k} \vec{F}_{ik} = \sum_{i} m_i \ddot{r}_i
\]
However, for each \(i\) and \(k\), \(\vec{F}_{ik} = -\vec{F}_{ki}\), so the double sum adds up to zero.

\[
\sum_{i} \vec{F}_i = \sum_{i} m_i \ddot{r}_i = M \ddot{A}
\]
That is, the acceleration of the Center of Mass is proportional to the net external force on the system. The Second “Law” reduces to that for a single particle of mass \(M\) located at \(\vec{R}\).

c. Momentum and energy
The total translational momentum of a system of particles is \(\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i = M \vec{V}\). At the same time, \(\frac{d\vec{P}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = M \ddot{A}\). Therefore, if the net external force on the system is zero, then the total translational momentum of the system is conserved.
By a similar token, the total kinetic energy of the system is 

\[ T = \sum_i \frac{1}{2} m_i \hat{\mathbf{r}}_i \cdot \hat{\mathbf{r}}_i \]

Now, we might rewrite this expression in terms of the particle velocities with respect to the center of mass, \( \hat{\mathbf{r}}_i = \mathbf{V} + \hat{\mathbf{r}}'_i \).

\[
T = \sum_i \frac{1}{2} m_i (\mathbf{V} + \hat{\mathbf{r}}'_i) \cdot (\mathbf{V} + \hat{\mathbf{r}}'_i) = \sum_i \frac{1}{2} m_i (\mathbf{V} \cdot \mathbf{V} + 2\mathbf{V} \cdot \hat{\mathbf{r}}'_i + \hat{\mathbf{r}}'_i \cdot \hat{\mathbf{r}}'_i)
\]

\[
T = \frac{1}{2} M \mathbf{V} \cdot \mathbf{V} + \sum_i \frac{1}{2} m_i \hat{\mathbf{r}}'_i \cdot \hat{\mathbf{r}}'_i
\]

We see the kinetic energy separated into two contributions: kinetic energy of the Center of Mass and kinetic energy with respect to the Center of Mass (or internal kinetic energy).

d. Angular momentum

The total angular momentum of a system of particles is the vector sum \( \mathbf{L} = \sum_i \mathbf{L}_i \). Of course, the individual angular momenta must be computed about the same axis point. For the sake of argument, let’s say that point is the origin: \( \mathbf{L} = \sum_i m_i \hat{\mathbf{r}}_i \times \hat{\mathbf{r}}_i \). Again, rewrite in terms of positions and velocities relative to the Center of Mass.

\[
\mathbf{L} = \sum_i m_i (\hat{\mathbf{r}} + \hat{\mathbf{r}}'_i) \times (\mathbf{V} + \hat{\mathbf{r}}'_i) = M \mathbf{R} \times \mathbf{V} + \sum_i m_i \hat{\mathbf{r}} \times \hat{\mathbf{r}}'_i + \sum_i m_i \hat{\mathbf{r}}'_i \times \mathbf{V} + \sum m_i \hat{\mathbf{r}}''_i \times \hat{\mathbf{r}}'_i
\]

The middle two terms are both zero, since \( \sum_i \hat{\mathbf{r}}'_i = \mathbf{0} \) and \( \sum_i \hat{\mathbf{r}}''_i = \mathbf{0} \).

\[
\mathbf{L} = M \mathbf{R} \times \mathbf{V} + \sum_i m_i \hat{\mathbf{r}}'_i \times \hat{\mathbf{r}}'_i
\]

Like the kinetic energy, the total angular momentum separates into two contributions: the angular momentum of the Center of Mass about the origin and the angular momentum about the Center of Mass. A similar analysis can be done if the axis point is not the origin.

If we consider the time rate of change of the angular momentum, we obtain

\[
\frac{d\mathbf{L}}{dt} = \sum_i m_i \hat{\mathbf{r}}_i \times \hat{\mathbf{r}}'_i + \sum_i m_i \hat{\mathbf{r}}'_i \times \hat{\mathbf{r}}_i = \mathbf{0} + \sum_i \hat{\mathbf{r}}_i \times (\mathbf{\ddot{F}}_i + \sum_k \mathbf{\ddot{F}}_{ik})
\]

\[
\frac{d\mathbf{L}}{dt} = \mathbf{\dot{N}} - \sum_i \sum_k \hat{\mathbf{r}}_{ik} \times \mathbf{\ddot{F}}_{ik},
\]

where the position of the \( k \)'th particle relative to the \( i \)'th particle is \( \hat{\mathbf{r}}_{ik} = \hat{\mathbf{r}}_k - \hat{\mathbf{r}}_i \). If the internal forces are central forces, then the double sum vanishes. If the external torque, \( \mathbf{\dot{N}} \), is zero, then the total angular momentum of the system is conserved.

2. Rocket

As an example of an object whose mass is changing as it moves, consider a rocket. We treat it as a reverse inelastic collision.

a. Impulse

Suppose the total mass of a moving object is not constant. Say the net external force acting on an object (such as a rocket or a rain drop) is \( \mathbf{F}_{ext} \). Assume that during a short time interval, \( \Delta t \), the \( \mathbf{F}_{ext} \) is approximately constant. Then the impulse delivered to the mass, \( m \), is
Further suppose that during that interval \( \Delta t \) the mass changes by an amount \( \Delta m \). The change in momentum that results is
\[
\Delta \dot{p} = m \ddot{v}_2 + \Delta m \dot{v}'_2 - (m + \Delta m) \ddot{v}_1.
\]
\[
\Delta \dot{p} = m (\ddot{v}_2 - \ddot{v}_1) + \Delta m (\dot{v}'_2 - \ddot{v}_1).
\]
We want to rewrite this in terms of the change in velocity of the mass, \( m \), and the relative velocity of the \( m \) and \( \Delta m \). Namely, \( \Delta \ddot{v} = \ddot{v}_2 - \ddot{v}_1 \) and \( \dot{V} = \dot{v}'_2 - \ddot{v}_2 \).
\[
\Delta \dot{p} = m \Delta \ddot{v} + \Delta m \dot{v}' + \Delta m \dot{V} = (m + \Delta m) \Delta \ddot{v} + \Delta m \dot{V}.
\]
The impulse, then, is \( \tilde{F}_{\text{ext}} \Delta t = (m + \Delta m) \Delta \ddot{v} + \Delta m \dot{V} \). We may as well just let \( m + \Delta m \) be \( m \) at this point.
\[
\tilde{F}_{\text{ext}} \Delta t = m \Delta \ddot{v} + \Delta m \dot{V}.
\]
Divide by \( \Delta t \);
\[
\tilde{F}_{\text{ext}} = m \frac{\Delta \ddot{v}}{\Delta t} + \frac{\Delta m}{\Delta t} \dot{V} \rightarrow m \frac{\ddot{v}}{dt} + \dot{V} \frac{dm}{dt}.
\]
Recap: \( \ddot{v} \) is the velocity of the object (rocket or rain drop), \( \dot{V} \) is the velocity of the \( \Delta m \) relative to the object, and \( \frac{dm}{dt} \) is the absolute value of the time rate of change in the mass of the object.
Actually, we have to be careful of the directions of things. As derived here, if \( \Delta m \) is leaving the object, then the object is losing mass and \( \Delta \ddot{v} \) is in the opposite direction as \( \dot{V} \). Consider a rocket in the absence of gravity or any other external force.
3. Collisions
Consider an isolated system of two particles, \( m_1 \) and \( m_2 \).

a. Reference frames
Lab frame—say the target is at rest,

\[
\frac{\vec{p}_1}{m_1} = \frac{\vec{p}_2}{m_2} \quad \text{and} \quad \theta = \phi_1 + \phi_2
\]

In general, this can be complicated, the more so if the particles are not point masses. We look at a special case.

If the collision is elastic \((Q = 0)\) and if \( m_1 = m_2 \), then \( p_1^2 = p_1'^2 + p_2'^2 \). On the other hand,

\[
p_1^2 = \vec{p}_1 \cdot \vec{p}_1 = (\vec{p}'_1 + \vec{p}'_2) \cdot (\vec{p}'_1 + \vec{p}'_2) = p_1'^2 + p_2'^2 + 2 \vec{p}'_1 \cdot \vec{p}'_2.
\]

Evidently, \( 2 \vec{p}'_1 \cdot \vec{p}'_2 = Q \) if \( m_1 = m_2 \). Further, if \( Q = 0 \), then \( \vec{p}'_1 \cdot \vec{p}'_2 = 0 \) which implies that \( \phi_1 + \phi_2 = \frac{\pi}{2} \). In order to solve for the out-going velocities, we need to be given one of the out-going angles. An alternative scenario is that \( \vec{p}'_1 = 0 \) and \( \vec{p}'_2 = \vec{p}_1 \).

Center-of-Mass frame—say the origin is at the Center of Mass.

By the definition of the Center of Mass,

\[
\vec{p}_1 + \vec{p}_2 = 0 = m_1 \vec{u}_1 + m_2 \vec{u}_2
\]

\[
\vec{p}_1' + \vec{p}_2' = 0 = m_1 \vec{u}_1' + m_2 \vec{u}_2'
\]
The energy equation is
\[ \frac{m_1^2 u_1^2}{2m_1} + \frac{m_2^2 u_2^2}{2m_2} = \frac{m_1^2 u_1'^2}{2m_1} + \frac{m_2^2 u_2'^2}{2m_2} + Q \]

Replace \( m_2^2 u_2^2 = m_1^2 u_1^2 \) and \( m_2^2 u_2'^2 = m_1^2 u_1'^2 \).
\[ \frac{m_1^2 u_1^2}{2m_1} + \frac{m_1^2 u_1^2}{2m_2} = \frac{m_1^2 u_1'^2}{2m_1} + \frac{m_2^2 u_2'^2}{2m_2} + Q \]
\[ \frac{m_1^2 u_1^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{m_1^2 u_1'^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + Q \]
\[ \frac{p_i^2}{2} \left( \frac{m_1 + m_2}{m_1 m_2} \right) = \frac{p_i'^2}{2} \left( \frac{m_1 + m_2}{m_1 m_2} \right) + Q \]

Define the reduced mass as \( \mu = \frac{m_1 m_2}{m_1 + m_2} \), in which case
\[ \frac{p_i^2}{2\mu} = \frac{p_i'^2}{2\mu} + Q. \]

Because the momenta of the two particles as measured in the Center of Mass frame are exactly opposite of each other, it is often more convenient to solve a collision problem in the Center of Mass frame. Of course, we would observe collisions in the Lab frame, so results must be transformed from one frame to the other, and back again.

b) Transforming from the Lab frame to the Center of Mass frame, and back again

\[ \begin{align*}
\vec{v}_{cm} &= \vec{v}_i' + \vec{v}_j' \\
\theta &= \theta' \\
\phi_i &= \phi_i' \\
\end{align*} \]

B. Rigid Body

A rigid body is a system of particles for which all the relative displacements between pairs of particles are fixed. That is, \( |\vec{r}_i - \vec{r}_j| = \) a constant for all \( i \) & \( j \).

1. Equations of motion

a. Rotation variables

Let \( \hat{k} \) be the axis of rotation, and consider a particle, or mass element, which is executing circular motion about the axis.
\[ R_i = \sqrt{x_i^2 + y_i^2} = \text{the distance from the axis of rotation,} \]
\[ \phi_i = \text{angular displacement,} \]
\[ \ddot{\omega} = \frac{d\phi_i}{dt} \hat{k} = \text{angular velocity, and} \]
\[ \nu_i = R_i \omega = \text{orbital speed} \]

By inspection, we can see that
\[ x_i = R_i \cos \phi_i \]
\[ \dot{x}_i = -R_i \sin \phi_i \frac{d\phi_i}{dt} = -R_i \omega \sin \phi_i \]
\[ y_i = R_i \sin \phi_i \]
\[ \dot{y}_i = R_i \omega \cos \phi_i \]
\[ \dot{z}_i = 0 \]

b. Kinetic energy
\[ T = \sum_i \frac{1}{2} m_i \nu_i^2 = \sum_i \frac{1}{2} m_i R_i^2 \omega^2 = \frac{\omega^2}{2} \sum_i m_i R_i^2 \]

We define the Moment of Inertia for the system as \( I = \sum_i m_i R_i^2 \), in terms of which the rotational kinetic energy can be written as \( T_{rot} = \frac{1}{2} I \omega^2 \). Notice that the numerical value of \( I \) depends on the location of the rotational axis!

c. Angular momentum
The total angular momentum of the system is \( \vec{L} = \sum \vec{r}_i \times m_i \nu_i \).

\[ \vec{L} = \sum_i \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ m_i & m_i & m_i \end{vmatrix} = \sum_i \{ \hat{k}(-m_i \dot{z}_i z_i) - \hat{j}(-m_i \dot{y}_i z_i) + \hat{i}(m_i \dot{y}_i x_i - m_i \dot{x}_i y_i) \} \]

We identify the components of the angular momentum vector as \( L_x = 0, \ L_y = 0, \) and \( L_z = \sum_i \{m_i \dot{y}_i x_i - m_i \dot{x}_i y_i\} \). Since the particles are going around in circles, the \( L_z \) can be written as \( L_z = \omega \sum_i m_i R_i^2 = I_z \omega \), where \( I_z \) means the moment of inertia calculated about the \( \hat{k} \)-axis.

d. Torque, or Moment of force
\[ \vec{\tau} = \sum \vec{r}_i \times \vec{F}_i \]

This is the total torque exerted on a system of particles. The \( \vec{F}_i \) is the net force acting on the \( i^{th} \) particle; the \( \vec{r}_i \) is the position of the \( i^{th} \) particle.
2. Computing moments of inertia

a. Discrete particles
The approach is to select an axis of rotation, then the moment of inertia about that axis is
\[ I = \sum_i R_i^2 m_i. \]
Note that the \( R_i^2 \) is the distance of the \( i^{th} \) particle from the axis of rotation.

b. Continuous mass distribution
If the distribution of the mass in a system is continuous, then the summation goes over to an integral over the mass density function.
\[ I = \int R^2 \rho \, dx \, dy \, dz \]
The whole of the integrand must be written in terms of spatial coordinates. The steps are
i) select the axis of rotation
ii) write \( R^2 \) in terms of coordinates
iii) write \( dm \) or \( \rho \) in terms of coordinates
iv) take advantage of symmetry.

example: a thin rod of length \( a \) and uniform mass density \( \rho \).

Because the rod is “thin” we have a one-dimensional problem. Let’s say the axis of rotation is the \( \hat{k} \) axis, while at the moment the rod lies along the \( \hat{i} \) axis. Then \( R = x \) and \( dm = \rho \, dx \).
\[ I = \int_0^a R^2 \, dm = \int_0^a x^2 \rho \, dx = \rho \frac{a^3}{3} \]

Suppose the mass density is not uniform, but instead \( \rho = 3x^2 - x \). Then the moment of inertia is
\[ I = \int_0^a R^2 \, dm = \int_0^a x^2 \rho \, dx = \int_0^a 3x^4 - x^3 \, dx = \frac{3}{5} a^5 - \frac{1}{4} a^4 \]
example: a disk of radius \( a \) and uniform mass density; the axis of rotation is perpendicular to the plane of the disk, through the center of the disk.
This is a two-dimensional problem, since the object is said to be a disk rather than a cylinder. The symmetry of the disk is circular, so polar coordinates are convenient.

\[ I = \int R^2 \, dm \]

\[ I = \int_0^{2\pi} \int_0^a r^2 \rho r \, dr \, d\theta = 2\pi\rho \int_0^a r^3 \, dr = 2\pi\rho \frac{a^4}{4} \]

This can be written in terms of the total mass, \( M \), of the disk by substituting \( \rho = \frac{M}{\pi a^2} \).

Of course, if the mass density is not uniform, then it is a function of \( r \) and \( \theta \).

c. Parallel axis theorem
Reconsider the thin rod, but now let the axis of rotation pass through the midpoint (i.e., the Center of Mass) of the rod rather than its endpoint. [Of course, if the mass distribution is not uniform, the Center of Mass is not at the rod’s midpoint.]

\[ I' = \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho x^2 \, dx = \rho \left[ \frac{x^3}{3} \right]_{-\frac{a}{2}}^{\frac{a}{2}} = \rho \frac{a^3}{12} \]

If \( I \) is the moment of inertia of the same rod about its endpoint, we can see that \( I = I' + \rho \frac{a^3}{4} \).

Once more in terms of the total mass of the rod, \( \rho \frac{a^3}{4} = M \frac{a^2}{4} \). But that’s just the moment of
inertia of a point mass at a distance \( R = \frac{a}{2} \) from the axis of rotation. Thus, what we have is the following statement: the moment of inertia of a system about a specified axis is equal to the moment of the system about a parallel axis through its Center of Mass, plus the moment of inertia of a point mass, equal to the total mass of the system, about the specified axis.

example: a hoop

The moment of inertia about an axis in the plane of the hoop and through its Center of Mass is

\[ I' = M \frac{a^2}{2} \]

The moment of inertia about a parallel axis at a distance \( r \) from the hoop’s Center of Mass is

\[ I = I' + Mr^2 = M \left( \frac{a^2}{2} + r^2 \right) \]

example: disk

\[ I = I' + Ma^2 = M \frac{a^2}{2} + Ma^2 = \frac{3}{2} Ma^2 \]

3. Laminar Motion of a Rigid Body

Laminar motion means that the translational motion of the rigid body is confined to a plane, say the \( xy \)-plane, while rotation occurs only around an axis perpendicular to that plane, say the \( z \)-axis.

a. Angular momentum

Imagine the rigid body is composed of many tiny mass elements, \( m_i \). The rotational equation of motion is

\[ \frac{d{\vec{L}}}{dt} = {\vec{N}} \]

\[ \frac{d}{dt} \left( \sum \vec{r}_i \times m_i \vec{\dot{r}}_i \right) = \sum \vec{r}_i \times \vec{F}_i \]

In terms of the Center of Mass,

\[ \frac{d}{dt} \sum \left[ (\vec{r}_{cm} + \vec{r}_i) \times m_i (\vec{\dot{r}}_{cm} + \vec{\dot{r}}_i) \right] = \sum (\vec{r}_{cm} + \vec{r}_i) \times \vec{F}_i \]

Now, \( \sum m_i \vec{r}_i = 0 \) and \( \sum m_i \vec{r}_i' = 0 \), so we have left
\[
\vec{r}_{cm} \times \sum m_i \frac{d\vec{r}_{cm}}{dt} + \frac{d}{dt} \sum \vec{r}_i \times m_i \ddot{r}_i' = \vec{r}_{cm} \times \sum \vec{F}_i + \sum \vec{r}_i' \times \vec{F}_i.
\]

However, \( \sum \vec{F}_i = \sum m_i \ddot{r}_i \) so the Center of Mass falls out completely.

\[
\frac{d}{dt} \sum \vec{r}_i' \times m_i \ddot{r}_i' = \sum \vec{r}_i' \times \vec{F}_i
\]

\[
\frac{d\vec{L}'}{dt} = \vec{N}'
\]

The time rate of change of angular momentum about the Center of Mass is equal to the net torque about the Center of Mass. Since this is true no matter what \( \ddot{r}_{cm} \) is, the translational and rotational motions can be treated separately.

b. Motion of a rigid body

example: rolling down an inclined plane

Consider a disk or a sphere or a hoop rolling down an inclined plane. The translational equation of motion for the Center of Mass is

\[ \vec{F} = M\ddot{r}_{cm} \]

For rotation about the Center of Mass the equation is

\[ \frac{d\vec{L}}{dt} = \frac{d}{dt} (I\ddot{\omega}) = \vec{N} \]

The component equations are

\[ M\ddot{x}_{cm} = Mg \sin \theta - F_f \]
\[ M\ddot{y}_{cm} = -Mg \cos \theta + F_N = 0 \]
\[ N = F_f a = I\ddot{\omega} \]

Note that rolling friction is not the same as sliding friction. If the object rolls without slipping, then \( \dot{x}_{cm} = a\omega \) and \( \dot{x}_{cm} = a\dot{\omega} \). Therefore the third equation becomes \( I \frac{\dot{x}_{cm}}{a} = F_f a \), which tells us that the frictional force must be \( F_f = I \frac{\dot{x}_{cm}}{a^2} \). Substitute this into the \( x \)-equation. . .
\[
M\ddot{x}_{cm} = Mg\sin\theta - I\ddot{x}_{cm}\frac{a^2}{a^2}
\]

\[
\ddot{x}_{cm} = \frac{g\sin\theta}{1 + \frac{I}{Ma^2}} = \text{a constant!}
\]

So, both \(\ddot{x}_{cm}\) and \(\omega\) are constants, when the object rolls down the incline without slipping.

If there is slipping, then \(\ddot{x}_{cm} \neq a\omega\). However, we do know that the sliding friction is \(F_f = \mu F_N\).

\[
M\ddot{x}_{cm} = Mg\sin\theta - F_f = Mg\sin\theta - \mu F_N = Mg\sin\theta - \mu Mg\cos\theta
\]

Integrate to obtain \(\dot{x}_{cm} = g(\sin\theta - \mu \cos\theta)\).

\[
I\ddot{\omega} = F_f a = \mu Mg a \cos\theta
\]

Integrate to obtain \(\omega = \frac{\mu Mga \cos\theta}{I} t\).

We find the relationship between the translational and rotational motions by dividing the two, thusly,

\[
\frac{\ddot{x}_{cm}}{\omega} = \frac{g(\sin\theta - \mu \cos\theta)}{\frac{\mu Mg a \cos\theta}{I} t} = \frac{I}{Ma}\left(\frac{\tan\theta}{\mu} - 1\right).
\]

e. Energy

For example, for the body rolling without slipping down the incline, the kinetic energy is
If we put the origin of coordinates at the top of the incline, then the gravitational potential energy is \( V = -M g x_{cm} \sin \theta \). Consequently, the total mechanical energy of the rolling object is

\[
E = \frac{1}{2} M \dot{x}_{cm}^2 + \frac{1}{2} I \omega^2 - M g x_{cm} \sin \theta = \frac{1}{2} \left( M + \frac{I}{a^2} \right) \dot{x}_{cm}^2 - M g x_{cm} \sin \theta = \text{constant}.
\]

When there is no slipping, the rolling frictional force does no work.

C. Central Forces

1. General Properties

\[ \mathbf{\ddot{r}} = F(r) \frac{\ddot{r}}{r} = \frac{F(r)}{r} \left( x i + y j + z k \right) \]

e.g., gravity: \( \mathbf{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^2} \). However, the force is not necessarily proportional to \( \frac{1}{r^2} \). For instance, the isotropic harmonic oscillator is \( \mathbf{F} = -k \mathbf{r} \). That’s a central force proportional to \( r \).

a. Conservative

Consider the curl of a central force; \( \nabla \times \mathbf{F} = \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{array} \right| \). The \( x \)-component is

\[
\left( \nabla \times \mathbf{F} \right)_x = \frac{\partial}{\partial y} F(r) \frac{z}{r} - \frac{\partial}{\partial z} F(r) \frac{y}{r} = z \frac{\partial}{\partial y} \left( \frac{F(r)}{r} \right) - y \frac{\partial}{\partial z} \left( \frac{F(r)}{r} \right) = z \frac{\partial^2 r}{\partial y \partial r} \frac{F(r)}{r} - y \frac{\partial^2 r}{\partial z \partial r} \frac{F(r)}{r}
\]

Now, \( \frac{\partial r}{\partial y} = \frac{\partial}{\partial y} \left( x^2 + y^2 + z^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left( x^2 + y^2 + z^2 \right)^{\frac{1}{2}} 2 y = \frac{y}{r} \). Similarly, \( \frac{\partial r}{\partial x} = \frac{x}{r} \) and \( \frac{\partial r}{\partial z} = \frac{z}{r} \).

Therefore, \( \left( \nabla \times \mathbf{F} \right)_x = \frac{zy}{r} \frac{\partial}{\partial r} F(r) - \frac{yz}{r} \frac{\partial}{\partial r} F(r) = 0 \). The same is true for the \( y \)- and \( z \)-components, so any central force is conservative, because we have not specified \( F(r) \).

b. Potential energy

The potential energy function is obtained from the work integral, if the force is known.

\[
W = \int_{r_0}^r \mathbf{F}(r) \cdot d\mathbf{r} = -\Delta V = -(V(r) - V(r_0)).
\]

On the other hand, if \( V(r) \) is known, the force components are obtained by

\[
\mathbf{F}(r) = -\nabla V(r) = \left( \frac{\dot{x}}{r} i + \frac{\dot{y}}{r} j + \frac{\dot{z}}{r} k \right) \frac{\partial V}{\partial r} = -\frac{\partial V}{r} \mathbf{r}.
\]
c. Angular momentum

The torque exerted by a central force about the origin is \( \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \). In a central force field, \( \vec{r} \times \vec{F} = 0 \); therefore the angular momentum is constant. That means both magnitude and direction. The motion is confined to a plane perpendicular to the angular momentum vector. As a result, we often use plane polar coordinates rather than Cartesian.

In polar coordinates, the velocity is \( \hat{r} = \hat{\theta} \times \vec{r} \). The angular momentum becomes

\[
\vec{L} = \vec{r} \times \vec{p} = \frac{d}{dt}(\vec{r} \times m(\vec{r} \hat{\theta} + r \hat{\theta})) = (mr^2 \hat{\theta}) \times \hat{\theta} = \text{constant vector}
\]

The constant magnitude of the angular momentum determines the orbits that are possible for a particle subject to any central force.

d. Equations of motion

We start, as we do every night, with Newton’s Second “Law.”

\[
\vec{F}(r) = \frac{d^2 \vec{r}}{dt^2} = m \frac{d^2 \vec{r}}{dt^2} + m \frac{d^2 \vec{r}}{dt^2} = m(\vec{r}' - r \hat{\theta} \hat{\theta}) + (r \hat{\theta} + 2 \hat{\theta} \hat{\theta}) \]

Decompose:

\[
\hat{r} : \quad \frac{d^2 \vec{r}}{dt^2} = \frac{d}{dt}(mr^2 \hat{\theta})
\]

\[
\hat{\theta} : \quad 0 = r \hat{\theta} + 2 \hat{\theta} \hat{\theta} = \frac{d}{dt}(mr^2 \hat{\theta})
\]

The angular equation says that the quantity \( mr^2 \hat{\theta} \) is a constant of the motion, namely the angular momentum. The radial equation includes the same centrifugal term that arose in the rotating reference frame.

e. Energy

The total mechanical energy is also a conserved quantity, or a constant of the motion.

\[
E = \frac{1}{2} m \hat{r} \cdot \hat{r} + V(r) = \frac{1}{2} \left( \hat{r}^2 + r^2 \hat{\theta}^2 \right) + V(r) = \frac{1}{2} m \hat{r}^2 + \frac{L^2}{2mr^2} + V(r)
\]

2. Orbits

a. Inverse square force

\[
\vec{F}(r) = \frac{K}{r^2} \hat{r}, \text{ where } K \text{ is a constant. The most familiar examples of an inverse square force are gravitation and the electrostatic force. The corresponding potential energy function is}
\]

\[
V(r) = \frac{K}{r}
\]

b. Trajectory

An orbit is just the trajectory of the particle, so in polar coordinates we are solving for \( r(\theta) \).
We start with the angular equations of motion:

\[ m\ddot{r} = \frac{K}{r^2} + mr \dot{\theta}^2 \]

and

\[ r^2 \ddot{\theta} = \frac{L}{m} = h = \text{constant}. \]

Solve the second equation for \( \ddot{\theta} \) and substitute into the first equation. \( \dot{\theta} = \frac{h}{r^2} \).

\[ m\ddot{r} = \frac{K}{r^2} + mr \frac{h^2}{r^4} = \frac{K}{r^2} + m \frac{h^2}{r^3} \]

\[ m\left( \ddot{r} - \frac{h^2}{r^3} \right) - \frac{K}{r^2} = 0 \]

Next, we need to express \( \frac{d^2r}{dt^2} \) in terms of \( \frac{d^2}{d\theta^2} \). To do this we start with the first derivative:

Let \( r = \frac{1}{u} \). Then

\[ \dot{r} = -\frac{1}{u^2} \dot{u} = -\frac{1}{u^2} \frac{d\theta}{dt} \frac{du}{d\theta} = -h \frac{du}{d\theta} \]

Take the derivative of this

\[ \ddot{r} = -h \frac{d}{dt} \left( \frac{du}{d\theta} \right) = -h \frac{d\theta}{dt} \frac{d}{d\theta} \left( \frac{du}{d\theta} \right) = -h^2 u^2 \frac{d^2u}{d\theta^2} \]

Substitute into the \( \ddot{r} \)-equation above.

\[ m \left( \frac{-h^2 u^2}{d\theta^2} \right) - h^2 u^2 - Ku^2 = 0 \]

\[ \frac{d^2u}{d\theta^2} + u = -\frac{K}{mh^2} = -\frac{mK}{L^2} = \text{constant}. \]

This is a differential equation with the form of a driven harmonic oscillator. The driving term is constant. The solution has the form \( u = A \cos(\theta - \theta_o) - \frac{mK}{L^2} \). Recalling that \( u = \frac{1}{r} \), we obtain

\[ r = \frac{1}{A \cos \theta - \frac{mK}{L^2}} \]

where we have set \( \theta_o = 0 \). This expression is the equation for a conic section whose general form is \( r = r_o \frac{1 + e}{1 + e \cos \theta} \), where \( e \) is the eccentricity and \( r_o \) is the closest approach to the origin. In this case \( e = -\frac{A l^2}{mK} \) and \( r_o = -\frac{L^2}{mK} \left( \frac{1}{1 + e} \right) = -\frac{1}{mK} \left( \frac{L^2}{1 + e} \right) = A \). The exact shape of the orbit depends on the values of the orbital parameters, \( e, r_o, \) and \( A \). These, in turn, are determined by the initial conditions through the total energy and angular momentum. The conic sections are: ellipse, parabola, and hyperbola.
c. Orbital parameters from initial conditions
We are interested in such things as the turning point(s) of the motion, the eccentricity of the orbit
and the orbital period if the orbit is closed.

The turning points are obtained from the total energy, just as is done with a harmonic oscillator.
\[ E = V(r) + \frac{1}{2} m \dot{r}^2 = \frac{K}{r} + \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} \]
At a turning point, \( \dot{r} = 0 \).
\[ \frac{L^2}{2m} \left( \frac{1}{r} \right)^2 + K \left( \frac{1}{r} \right) - E = 0 \]
Solve for \( \frac{1}{r} \) using the quadratic formula—obtain two roots.
\[ \frac{1}{r_o} = -\frac{Km}{L^2} + \left( \frac{Km}{L^2} \right)^2 + \frac{2mE}{L^2} \]
\[ \frac{1}{r_1} = -\frac{Km}{L^2} - \left( \frac{Km}{L^2} \right)^2 + \frac{2mE}{L^2} \]
Notice that if \( K > 0 \) (the force is repulsive), then \( r_1 < 0 \) doesn’t exist and there is only one turning point—the orbit is open.

We had previously that \( \frac{1}{r_o} = -\left( \frac{Km}{L^2} - A \right) \), where \( A \) was the amplitude of the solution to the
differential equation for \( r \) in terms of \( \theta \). Set this equal to the expression above and solve for \( A \).
\[ A = \left[ \frac{Km}{L^2} \right]^2 + \frac{2mE}{L^2} \]
Plug this into the expression for the eccentricity
\[ e = \frac{AL^2}{Km} = \left[ 1 + \frac{2EL^2}{K^2m} \right]^{\frac{1}{2}} \]
Now, given \( E \) and \( L \), we can obtain \( r_o, r_1, \) and \( e \).

The period of an orbit has meaning only for a closed orbit—an ellipse. For an ellipse, the total
area enclosed by the orbit is \( A = \pi a^2 \sqrt{1 - e^2} = \frac{2\pi a^2 EL^2}{K^2m} \), where \( a \) is the semimajor axis. On the
other hand, the total area swept out by the \( \vec{r} \) vector is also 
\[
A = \int_0^A dA' = \int_0^t \frac{L}{2m} \, dt = \frac{L \tau}{2m}.
\]
So set ‘em equal and solve for the period, \( \tau \). Using also the fact that 
\[
E = \frac{1}{2} K^2,
\]
we get
\[
\tau = \frac{4 \pi a^2}{K^2} = \frac{\pi L}{2 \sqrt{E K}}.
\]
Commonly, we start with \( \vec{r}_o \) perpendicular to \( \vec{r}_o' \) at \( \theta_o = 0 \) and \( \left| \vec{r}_o' \right| = v_o \). Then the angular momentum and total energy are determined: 
\[
L = mr_o v_o \quad \text{and} \quad E = \frac{K}{r_o} + \frac{L^2}{2mr_o^2} = \frac{K}{r_o} + \frac{1}{2} m v_o'^2.
\]
Notice that the origin is not at the center of the ellipse, but at one of the foci.